Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

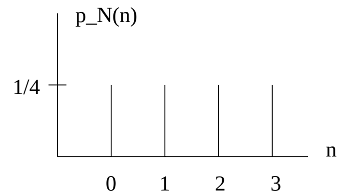
6.041/6.431: Probabilistic Systems Analysis

(Spring 2005)

Problem Set 4 Solutions
Due: March 2, 2005

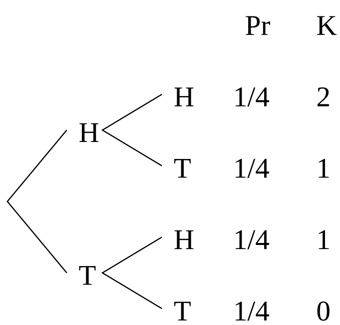
1. (a)

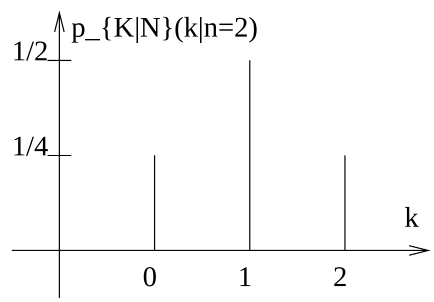
The first part can be completed without reference to anything other than the die roll:



(b)

Given that we flip the coin twice, a sequential sample space for this experiment is as follows. We notice that we get 2 heads with probability 1/4, 1 head with probability $1/4 + 1/4 = 1/2$, and 0 head with probability 1/4.





(c) The remaining parts can be done by straightforward but tedious collection of the relevant events in the original sample space. For example,

$$p_{N|K}(2 \mid 2) = \frac{\mathbf{P}(N = 2,\, K = 2)}{\mathbf{P}(K = 2)}$$

where

$$\mathbf{P}(N = 2,\, K = 2) = \mathbf{P}(K = 2|N = 2)\mathbf{P}(N = 2) = (1/4)(1/4) = 1/16$$

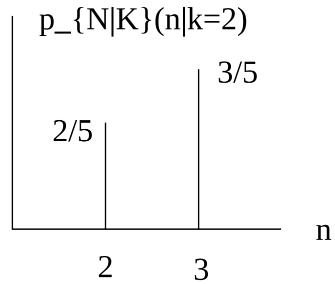
and by using the total probability theorem

$$\begin{aligned}
\mathbf{P}(K = 2) &= \sum_{n=0}^{3} \mathbf{P}(K = 2|N = n)\mathbf{P}(N = n) = (1/4)\sum_{n=0}^{3} \mathbf{P}(K = 2|N = n) \\
&= (1/4)(0 + 0 + 1/4 + \binom{3}{2}(1/2)^3) = 1/16 + 3/32
\end{aligned}$$

so we obtain

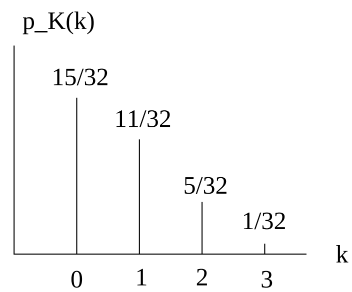
$$p_{N|K}(2 \mid 2) = \frac{1/16}{1/16 + 3/32} = 2/5.$$

The remaining terms of the PMF are computed in the same way.



(d) For example,

$$\begin{aligned} p_K(0) &= \mathbf{P}(K=0) = \sum_{n=0}^{3} \mathbf{P}(K=0|N=n)\mathbf{P}(N=n) = (1/4)\sum_{n=0}^{3} \mathbf{P}(K=0|N=n) \\ &= (1/4)(1+1/2+1/4+1/8) = 15/32. \end{aligned}$$



(e) Let A denote the event that K is an odd number. The first term of the PMF for N conditional on A is computed as

$$p_{N|A}(0) = \frac{\mathbf{P}(\{N=0\}\cap A)}{\mathbf{P}(A)} = \frac{\mathbf{P}(N=0,\, K \text{ odd})}{\mathbf{P}(K \text{ odd})} = 0$$

since K must be at least 1 to be an odd number, which is impossible if $N = 0$. Similarly,

$$p_{N|A}(1) = \frac{\mathbf{P}(\{N=1\}\cap A)}{\mathbf{P}(A)} = \frac{\mathbf{P}(N=1,\, K \text{ odd})}{\mathbf{P}(K \text{ odd})} = \frac{\mathbf{P}(N=1,\, K=1)}{\mathbf{P}(K \text{ odd})}.$$

We have

$$\mathbf{P}(N=1,\, K=1) = \mathbf{P}(K=1|N=1)\mathbf{P}(N=1) = (1/2)(1/4) = 1/8$$

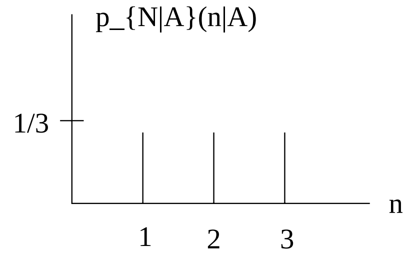
and

$$\mathbf{P}(K \text{ odd}) = p_K(1) + p_K(3) = 11/32 + 1/32 = 12/32 = 3/8$$

so

$$p_{N|A}(1) = 1/3.$$

The other terms are obtained similarly.



(f)
$\mathbf{E}[N] = (1/4)(0+1+2+3) = 1.5$,
$\text{var}[N] = (1/4)((0-1.5)^2 + (1-1.5)^2 + (2-1.5)^2 + (3-1.5)^2) = 1.25$
$\mathbf{E}[K|N=2] = (1/4)(0) + (1/2)(1) + (1/4)(2) = 1$,
$\text{var}[K|N=2] = (1/4)(0-1)^2 + (1/2)(1-1)^2 + (1/4)(2-1)^2 = 0.5$
$\mathbf{E}[N|K=2] = (2/5)(2) + (3/5)(3) = 2.6$,
$\text{var}[N|K=2] = (2/5)(2-2.6)^2 + (3/5)(3-2.6)^2 = 0.24$
$\mathbf{E}[K] = (15/32)(0) + (11/32)(1) + (5/32)(2) + (1/32)(3) = 0.75$,
$\text{var}[K] = (15/32)(0-0.75)^2 + (11/32)(1-0.75)^2 + (5/32)(2-0.75)^2 + (1/32)(3-0.75)^2 = \frac{11}{16}$
$\mathbf{E}[N|K=odd] = (1/3)(1+2+3) = 2$,
$\text{var}[N|K=odd] = (1/3)((1-2)^2 + (2-2)^2 + (3-2)^2) = \frac{2}{3}$

2. (a) From the joint PMF, there are six (x, y) coordinate pairs with nonzero probabilities of occurring. These pairs are (1, 1), (1, 3), (2, 1), (2, 3), (4, 1), and (4, 3). The probability of a pair is proportional to the sum of the x^2 and y^2 coordinate of the pair. Because the probability of the entire sample space must equal 1, we have:

$$(1+1)c + (1+9)c + (4+1)c + (4+9)c + (16+1)c + (16+9)c = 1.$$

Solving for c, we get $c = \boxed{\frac{1}{72}}$

(b) There are three sample points for which $y < x$.

$$\mathbf{P}(Y < X) = \mathbf{P}\{(2,1)\} + \mathbf{P}\{(4,1)\} + \mathbf{P}\{(4,3)\} = \frac{5}{72} + \frac{17}{72} + \frac{25}{72} = \boxed{\frac{47}{72}}$$

(c) There are two sample points for which $y > x$.

$$\mathbf{P}(Y > X) = \mathbf{P}\{(1,3)\} + \mathbf{P}\{(2,3)\} = \frac{10}{72} + \frac{13}{72} = \boxed{\frac{23}{72}}$$

(d) There is only one sample point for which $y = x$.

$$\mathbf{P}(Y = X) = \mathbf{P}\{(1,1)\} = \boxed{\frac{2}{72}}$$

Notice that, using the above two parts:

$$\mathbf{P}(Y < X) + \mathbf{P}(Y > X) + \mathbf{P}(Y = X) = \frac{47}{72} + \frac{23}{72} + \frac{2}{72} = 1$$

as expected.

(e) There are three sample points for which $y = 3$.

$$\mathbf{P}(Y = 3) = \mathbf{P}\{(1,3)\} + \mathbf{P}\{(2,3)\} + \mathbf{P}\{(4,3)\} = \frac{10}{72} + \frac{13}{72} + \frac{25}{72} = \boxed{\tfrac{48}{72}}.$$

(f) In general, for two discrete random variable X and Y for which a joint PMF is defined, we have:

$$p_X(x) = \sum_{y=-\infty}^{\infty} p_{X,Y}(x,y)$$

and

$$p_Y(y) = \sum_{x=-\infty}^{\infty} p_{X,Y}(x,y)$$

In this problem the range of X and Y is quite restricted so we can determine the marginal PMFs by enumeration. For example,

$$p_X(2) = \mathbf{P}\{(2,1)\} + \mathbf{P}\{(2,3)\} = \frac{18}{72}.$$

Overall, we get:

$$p_X(x) = \begin{cases} 12/72 & , \quad x = 1 \\ 18/72 & , \quad x = 2 \\ 42/72 & , \quad x = 4 \\ 0 & , \quad \text{otherwise} \end{cases} .$$

$$p_Y(y) = \begin{cases} 24/72 & , \quad y = 1 \\ 48/72 & , \quad y = 3 \\ 0 & , \quad \text{otherwise} \end{cases} .$$

(g) In general, the expected value of any discrete random variable X equals:

$$\mathbf{E}[X] = \sum_{x=-\infty}^{\infty} x p_X(x)$$

For this problem,

$$\mathbf{E}[X] = 1\frac{12}{72} + 2\frac{18}{72} + 4\frac{42}{72} = \boxed{3}$$

and

$$\mathbf{E}[Y] = 1\frac{24}{72} + 3\frac{48}{72} = \boxed{\tfrac{7}{3}}$$

(h) The variance of a random variable X can be computed as $\mathbf{E}[X^2] - \mathbf{E}[X]^2$ or as $\mathbf{E}[(X - \mathbf{E}[X])^2]$. We use the second approach here because X and Y take on such limited ranges.

$$\text{var}(X) = (1-3)^2\frac{12}{72} + (2-3)^2\frac{18}{72} + (4-3)^2\frac{42}{72} = \boxed{\frac{3}{2}}$$

$$\text{var}(Y) = (1-\frac{7}{3})^2\frac{24}{72} + (3-\frac{7}{3})^2\frac{48}{72} = \boxed{\frac{8}{9}}$$

3. Let random variable X be the number of trials you need to open the door, and define K_i to be the event that the ith key selected opens the door.

   (a) We have

$$\begin{aligned} p_X(1) &= \mathbf{P}(K_1) = \frac{1}{5} \\ p_X(2) &= \mathbf{P}(K_1^c)\mathbf{P}(K_2 \mid K_1^c) = \left(\frac{4}{5}\right)\left(\frac{1}{4}\right) = \frac{1}{5} \\ p_X(3) &= \mathbf{P}(K_1^c)\mathbf{P}(K_2^c \mid K_1^c)\mathbf{P}(K_3 \mid K_1^c \cap K_2^c) = \left(\frac{4}{5}\right)\left(\frac{3}{4}\right)\left(\frac{1}{3}\right) = \frac{1}{5} \end{aligned}$$

   Proceeding as such, we see that the PMF for X is

$$p_X(x) = \begin{cases} \frac{1}{5} & x = 1,2,3,4,5 \\ 0 & \text{otherwise} \end{cases}$$

   We can also view the problem as ordering the keys in advance and then trying them in succession, in which case the probability of any of the five keys being correct is 1/5. Since we have a discrete uniform distribution, the mean and variance can be readily determined:

$$\mathbf{E}[X] = \frac{a+b}{2} = \frac{1+5}{2} = \boxed{3}, \qquad \text{var}(X) = \frac{(b-a)(b-a+2)}{12} = \frac{(5-1)(5-1+2)}{12} = \boxed{2}.$$

   (b) In this case, X is a geometric random variable with $p = 1/5$. The mean and variance are then:

$$\mathbf{E}[X] = \frac{1}{p} = \frac{1}{1/5} = \boxed{5}, \qquad \text{var}(X) = \frac{1-p}{p^2} = \frac{1-1/5}{(1/5)^2} = \boxed{20}.$$

4. Let us consider a decision in which the contestant must choose between answering question 1 or question 2 first.

   If he attempts question 1 first, then he will win:

$$\begin{gathered} 0 \text{ with probability } 1-p_1 \\ v_1 \text{ with probability } p_1(1-p_2) \\ v_1 + v_2 \text{ with probability } p_1 p_2 \end{gathered}$$

For this case, the expected money won $\mathbf{E}[w1]$ is given by:

$$\mathbf{E}[w1] = 0(1-p_1) + v_1 p_1 (1-p_2) + (v_1+v_2) p_1 p_2$$

Similarly, if she attempts question 2 first, then she will win:

$$\begin{gathered} 0 \text{ with probability } 1-p_2 \\ v_2 \text{ with probability } p_2(1-p_1) \\ v_1+v_2 \text{ with probability } p_1 p_2 \end{gathered}$$

For this case, the expected money won $\mathbf{E}[w2]$ is given by:

$$\mathbf{E}[w2] = 0(1-p_2) + v_2 p_2 (1-p_1) + (v_1+v_2) p_1 p_2$$

Therefore, it is better to try question 1 first if

$$\mathbf{E}[w1] > \mathbf{E}[w2]$$

$$v_1 p_1 (1-p_2) > v_2 p_2 (1-p_1)$$