Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Spring 2005)

Tutorial 9
Week of April 11, 2005

1. (a) Let R be the total number of messages received during an interval of duration t. R is a Poisson RV with average arrival rate $\lambda_A + \lambda_B$. Therefore,

$$\begin{aligned} &\mathbf{P}(\text{exactly nine messages in time interval } t) \\ &= p_R(9) \\ &= \frac{((\lambda_A + \lambda_B)t)^9 e^{-(\lambda_A+\lambda_B)t}}{9!} \end{aligned}$$

(b) Let R be defined as in part a. Then,

$$N = W_1 + \cdots + W_R$$

We see that N is a random sum of random variables. Therefore,

$$\begin{aligned} \mathbf{E}[N] &= \mathbf{E}[W]\mathbf{E}[R] \\ &= \left(1 \cdot \frac{2}{6} + 2 \cdot \frac{3}{6} + 3 \cdot \frac{1}{6}\right)(\lambda_A + \lambda_B)t \\ &= \frac{11}{6}(\lambda_A + \lambda_B)t \end{aligned}$$

(c) Three-word messages arrive from transmitter A in a Poisson manner with average arrival rate $\lambda_A p_W(3)$. Therefore, X is $8th$ order Erlang and,

$$f_X(x) = \frac{\left(\frac{1}{6}\lambda_A\right)^8 x^7 e^{-\frac{1}{6}\lambda_A x}}{7!}$$

(d) Damage to a particular word is independent of what happens to all other words. Thus, picking a damaged word at random and determining the length of the message it came from should be the same as picking any random word (damaged or not damaged) and determining the length of the message it came from. The latter problem is more recognizable as a random incidence problem. Hence we can write the pmf of a random variable K, which describes the length of a message associated with a randomly selected word:

$$p_K(k) = \begin{cases} 2/11 & , \quad k = 1 \\ 6/11 & , \quad k = 2 \\ 3/11 & , \quad k = 3 \\ 0 & , \quad \text{otherwise} \end{cases}$$

where we obtained the pmf from the formula:

$$p_K(k) = \frac{k p_W(k)}{\mathbf{E}[W]}$$

The probability in question is 3/11.

Note that we can also obtain this result using Bayes' rule.

$$
\begin{aligned}
&\mathbf{P}(\text{in a message with 3 words} \mid \text{the word is damaged}) \\
&= \frac{\mathbf{P}(\text{the word is damaged} \mid \text{in a message with 3 words}) \cdot \mathbf{P}(\text{in a message with 3 words})}{\mathbf{P}(\text{the word is damaged})} \\
&= \frac{\mathbf{P}(\text{the word is damaged}) \cdot \mathbf{P}(\text{in a message with 3 words})}{\mathbf{P}(\text{the word is damaged})} \\
&= 3/11,
\end{aligned}
$$

where the second equality comes from the independence of damaged words and the number of words per message.

(e) Every message received either came from transmitter A or transmitter B. So, each message is a Bernoulli trial. We will say a success has occurred if a message that we receive comes from transmitter A. The probability of success in one of the Bernoulli trials is $\lambda_A/(\lambda_A + \lambda_B)$. The number of successes in a series of independent Bernoulli trials is a Binomial RV. Therefore, the probability that exactly eight of the next twelve messages received will be from transmitter A is,

$$\binom{12}{8}\left(\frac{\lambda_A}{\lambda_A+\lambda_B}\right)^8\left(\frac{\lambda_B}{\lambda_A+\lambda_B}\right)^4$$

2. (a) We may view the time until a particular player is injured as the time until the first arrival in a Poisson process of rate λ. Since each player is independent, and since we have 8 players, we have 8 independent Poisson processes of rate λ. Thus, we may view the time until any player is injured as the time until the first arrival in the merged Poisson process, which has rate 8λ. The expected time till the first arrival is therefore

$$\frac{1}{8\lambda}$$

(b) If we assume that you can only be injured if you are playing, then the time till the first injury is exponential with rate $8 \cdot \lambda$. Thus we have an Erlang of order $n - 7$, rate 8λ.

$$f_{T_{n-7}}(t) = \begin{cases} \frac{(8\lambda)^{n-7} t^{(n-7)-1} e^{-8\lambda t}}{((n-7)-1)!} & t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$