LECTURE 7

- **Readings:** Finish Chapter 2

Lecture outline

- Review of joint PMFs
- More expectations
- Binomial distribution revisited
- The hat problem

Review

$$p_X(x) = \mathbf{P}(X = x)$$

$$p_{X,Y}(x,y) = \mathbf{P}(X = x, Y = y)$$

$$p_{X|Y}(x \mid y) = \mathbf{P}(X = x \mid Y = y)$$

$$p_X(x) = \sum_y p_{X,Y}(x,y)$$

$$p_{X,Y}(x,y) = p_X(x) p_{Y|X}(y \mid x)$$

Independent random variables

$$p_{X,Y,Z}(x,y,z) = p_X(x) p_{Y|X}(y \mid x) p_{Z|X,Y}(z \mid x,y)$$

- Random variables X, Y, Z are independent if:

  $$p_{X,Y,Z}(x,y,z) = p_X(x) \cdot p_Y(y) \cdot p_Z(z)$$

  for all x, y, z

y \ x	1	2	3	4
4	1/20	2/20	2/20	
3	2/20	4/20	1/20	2/20
2		1/20	3/20	1/20
1		1/20		

- Independent?
- What if we condition on $X \leq 2$ and $Y \geq 3$?

Expectations

$$\mathbf{E}[X] = \sum_x x p_X(x)$$

$$\mathbf{E}[g(X,Y)] = \sum_x \sum_y g(x,y) p_{X,Y}(x,y)$$

- In general: $\mathbf{E}[g(X,Y)] \neq g(\mathbf{E}[X], \mathbf{E}[Y])$

- $\mathbf{E}[\alpha X + \beta] = \alpha \mathbf{E}[X] + \beta$
- $\mathbf{E}[X + Y + Z] = \mathbf{E}[X] + \mathbf{E}[Y] + \mathbf{E}[Z]$

- If X, Y are independent:
  - $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$
  - $\mathbf{E}[g(X)h(Y)] = \mathbf{E}[g(X)] \cdot \mathbf{E}[h(Y)]$

Variances

- Var$(aX) = a^2$Var(X)
- Var$(X + a) =$ Var(X)
- Let $Z = X + Y$.
  If X, Y are independent:

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$$

- Examples:
  - If $X = Y$, Var$(X + Y) =$
  - If $X = -Y$, Var$(X + Y) =$
  - If X, Y indep., and $Z = X - 3Y$, Var$(Z) =$

Binomial mean and variance

- X = # of successes in n independent trials
  - probability of success p

$$E[X] = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

- $X_i = \begin{cases} 1, & \text{if success in trial } i, \\ 0, & \text{otherwise} \end{cases}$
- $\mathbf{E}[X_i] =$
- $\mathbf{E}[X] =$
- Var$(X_i) =$
- Var$(X) =$

The hat problem

- n people throw their hats in a box and then pick one at random.
  - X: number of people who get their own hat
  - Find $\mathbf{E}[X]$

$$X_i = \begin{cases} 1, & \text{if } i \text{ selects own hat} \\ 0, & \text{otherwise.} \end{cases}$$

- $X = X_1 + X_2 + \cdots + X_n$
- $\mathbf{P}(X_i = 1) =$
- $\mathbf{E}[X_i] =$
- Are the X_i independent?
- $\mathbf{E}[X] =$

Variance in the hat problem

- Var$(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = \mathbf{E}[X^2] - 1$

$$X^2 = \sum_i X_i^2 + \sum_{i,j: i \neq j} X_i X_j$$

- $\mathbf{E}[X_i^2] =$

$$\mathbf{P}(X_1 X_2 = 1)$$

$$= \mathbf{P}(X_1 = 1) \cdot \mathbf{P}(X_2 = 1 \mid X_1 = 1) =$$

- $\mathbf{E}[X^2] =$
- Var$(X) =$