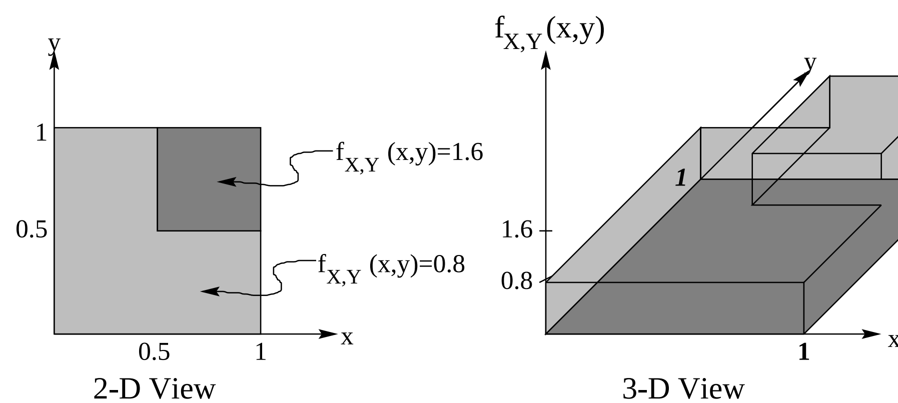
Problem Set 5
Due: March 16, 2005

1. An ATM is replenished with cash every day at 12 p.m. The amount withdrawn (in hundreds of thousands of dollars) in the 24 hours between refills behaves like a random variable with probability density function:

$$f_Z(z) = \begin{cases} 5(1-z)^4 & 0 < z < 1 \\ 0 & \text{otherwise} \end{cases}$$

How much money should the bank deposit in the ATM so it can be 99% confident that the supply will last the 24 hours?

2. Continuous random variables X and Y each take on experimental values between zero and one, with the joint pdf indicated below (the cutoff between probability density 0.8 and 1.6 occurs at $x = 0.5$ and $y = 0.5$):



(a) Are X and Y independent? Present a convincing argument for your answer.

(b) Prepare neat, fully labelled plots for $f_X(x)$ and $f_{Y|X}(y|0.75)$.

(c) Let $R = XY$ and let A be the event $X < 0.5$. Evaluate $\mathbf{E}[R|A]$.

(d) Let $W = \min\{X, Y\}$ and determine the correct expression for the cumulative distribution function (CDF) for W for all w. You should be able to reason out this part without doing any formal integrals.

3. Suppose that a binary message – either 0 or 1 – must be transmitted by wire from location A to location B. However, the data sent over the wire are subject to a channel noise disturbance, so to reduce the possibility of error, the value 2 is sent over the wire when the message is 1, and the value -2 is sent when the message is 0. If X, $X = \{-2, 2\}$, is the value sent at location A, the value received at location B, denoted as R, is given by

$$R = X + N \ ,$$

where N is the channel noise disturbance, which is independent of X. When the message is received at location B, the receiver decodes it according to the following rule:

if $R \geq .5$, then conclude that message 1 was sent
if $R < .5$, then conclude that message 0 was sent

Assuming that the channel noise, N, is a unit normal random variable and that the message 0 or 1 is sent with equal probability, what is the probability that we conclude that the wrong message was sent? This is the probability of error for this communication channel.

4. Random variables X and Y have the joint PDF of

$$f_{X,Y}(x,y) = \begin{cases} 6x & , \quad x > 0, y > 0, x + y \leq 1.0 \\ 0 & , \quad \text{otherwise} \end{cases}$$

   (a) Obtain an expression for and sketch the PDF of $f_Y(y)$.
   (b) Determine the conditional expectation and variance for X given $Y = 0.5$.

5. Beginning at time $t = 0$ we begin using bulbs, one at a time, to illuminate a room. Bulbs are replaced immediately upon failure. Each new bulb is selected independently by an equally likely choice between a Type-A bulb and a Type-B bulb.

   The lifetime, X, of any particular bulb of a particular type is an independent random variable with the following PDF:

$$\text{For Type-A Bulbs:} \quad f_X(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$\text{For Type-B Bulbs:} \quad f_X(x) = \begin{cases} 3e^{-3x} & x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

   (a) Find the expected time until the first failure.
   (b) Find $\mathbf{P}(D)$, the probability that there are no bulb failures during the first τ hours of this process.
   (c) Given that there are no failures during the first τ hours of this process, determine $\mathbf{P}(T_{1A}|D)$, the conditional probability that the first bulb used is a Type-A bulb.
   (d) Given that there are no failures during the first τ hours of this process, determine the total expected time until the first failure (i.e., the expected time elapsed from $t = 0$ until the first bulb fails).