

**6.041 Quiz 1 Solutions**  
**March 7, 2004**

**Problem 1:** (27 points)

- (a) (15 points) Yes. To see this, let  $X$  be a random variable that takes values 1 or 0 depending on whether the first coin came out head or tail, respectively. Similarly, let  $Y$  be a random variable that takes values 1 or 0 depending on whether the second coin came out head or tail, respectively. Finally, let  $W$  be a random variable that takes values 1 or 0 depending on whether you win or lose, respectively.

We have  $p_{W|X}(0|0) = p_Y(1) = 1/2$  and  $p_{W|X}(0|1) = p_Y(0) = 1/2$ . Therefore,

$$p_W(0) = p_{W|X}(0|0)p_X(0) + p_{W|X}(0|1)p_X(1) = (1/2)(1/2) + (1/2)(1/2) = 1/2.$$

Likewise,  $p_{W|X}(1|0) = p_Y(0) = 1/2$  and  $p_{W|X}(1|1) = p_Y(1) = 1/2$ . Therefore,

$$p_W(1) = p_{W|X}(1|0)p_X(0) + p_{W|X}(1|1)p_X(1) = (1/2)(1/2) + (1/2)(1/2) = 1/2.$$

Hence, we see that  $p_W(w) = p_{W|X}(w|x)$  for all  $w, x = 0, 1$ , which establishes the independence of  $X$  and  $W$ .

Showing that  $Y$  and  $W$  are independent is completely symmetric.

- (b) (12 points) No. To see this, note that once we know we have won, if in addition we were to know that the second coin came up heads, then we know that the first coin also came up heads. More formally, note that we have

$$p_{X|Y,W}(1|1,1) = 1,$$

but, since  $X$  and  $W$  are independent,

$$p_{X|W}(1|1) = p_X(1) = 1/2.$$

**Problem 2:** (70 points)

- (a) (10 points) One quarter is drawn from Jerome's left pocket. The probability that it is the Canadian one is  $1/5$ , so Jerome's Canadian quarter remains in his left pocket with probability  $4/5$ .
- (b) (18 points) Let  $U$  be the number of Canadian quarters remaining in Jerome's left pocket, and let  $V$  be the number of American ones. Hence  $p_U(0) = 1/5$  and  $p_U(1) = 4/5$ . Therefore,  $\mathbf{E}[U] = \mathbf{E}[U^2] = 4/5$ , so  $\text{var}(U) = 4/5 - (4/5)^2 = 4/25$ . Now,  $V = 4 - U$  and it follows that

$$\mathbf{E}[V] = 4 + \mathbf{E}[-U] = 4 - 4/5 = 16/5,$$

and that

$$\text{var}(V) = \text{var}(4) + \text{var}(-U) = \text{var}(U) = 4/25.$$

- (c) (10 points) The question asks for the probability of three successes in five independent Bernoulli trials, where a success here is a draw from Jerome's left pocket. Therefore, the probability in question is

$$\binom{5}{3} p^3 (1-p)^2.$$

- (d) (12 points) The eighth quarter drawn is the fifth drawn from Jerome's left pocket if and only if four of the first seven draws are from his left pocket and the eighth is from his left pocket. Therefore, the probability in question is

$$\binom{7}{4} p^4 (1-p)^3 \cdot p = \binom{7}{4} p^5 (1-p)^3.$$

- (e) (20 points) From part (d), we have  $\mathbf{P}(X = 2 \cap A) = \binom{7}{4} p^5 (1-p)^3$ . More generally, we see that, for  $x = 1, 2, \dots, 5$ , the event  $X = x \cap A$  is equivalent to the event that the  $(10-x)$ th quarter drawn is the fifth drawn from Jerome's left pocket. Hence,

$$\mathbf{P}(X = x \cap A) = \binom{9-x}{4} p^4 (1-p)^{5-x} \cdot p = \binom{9-x}{4} p^5 (1-p)^{5-x}$$

for  $x = 1, 2, \dots, 5$ .