Recitation 18: Solutions
April 21, 2005
Markov Processes (Section 6.1-6.2)

1. (a) Let A_k be the event that the process enters S_2 for first time on trial k. The only way to enter state S_2 for the first time on the kth trial is to enter state S_3 on the first trial, remain in S_3 for the next $k-2$ trials, and finally enter S_2 on the last trial. Thus,

$$\mathbf{P}(A_k) = p_{03} \cdot p_{33}^{k-2} \cdot p_{32} = \left(\frac{1}{3}\right)\left(\frac{1}{4}\right)^{k-2}\left(\frac{1}{4}\right) = \frac{1}{3}\left(\frac{1}{4}\right)^{k-1} \quad \text{for} \quad k = 2, 3, \ldots$$

(b) Let A be the event that the process never enters S_4.
There are three possible ways for A to occur. The first two are if the first transition is either from S_0 to S_1 or S_0 to S_5. This occurs with probability $\frac{2}{3}$. The other is if The first transition is from S_0 to S_3, and that the next change of state *after* that is to the state S_2. We know that the probability of going from S_0 to S_3 is $\frac{1}{3}$. Given this has occurred, and given a change of state occurs from state S_3, we know that the probability that the state transitioned to is the state S_2 is simply $\frac{\frac{1}{4}}{\frac{1}{4}+\frac{1}{2}} = \frac{1}{3}$. Thus, the probability of transitioning from S_0 to S_3 and then eventually transitioning to S_2 is $\frac{1}{9}$. Thus, the probability of never entering S_4 is $\frac{2}{3} + \frac{1}{9} = \frac{7}{9}$.

(c) $\mathbf{P}(\{\text{process enters } S_2 \text{ and then leaves } S_2 \text{ on next trial}\})$

$$\begin{aligned}
&= \mathbf{P}(\{\text{process enters } S_2\})\mathbf{P}(\{\text{leaves } S_2 \text{ on next trial }\}|\{\text{ in } S_2\}) \\
&= \left[\sum_{k=2}^{\infty} \mathbf{P}(A_k)\right] \cdot \frac{1}{2} \\
&= \left[\sum_{k=2}^{\infty} \frac{1}{3}(\frac{1}{4})^{k-1}\right] \cdot \frac{1}{2} \\
&= \frac{1}{6} \cdot \frac{\frac{1}{4}}{1-\frac{1}{4}} \\
&= \frac{1}{18}.
\end{aligned}$$

(d) This event can only happen if the sequence of state transitions is as follows:

$$S_0 \longrightarrow S_3 \longrightarrow S_2 \longrightarrow S_1.$$

Thus, $\mathbf{P}(\{\text{process enters } S_1 \text{ for first time on third trial}\}) = p_{03} \cdot p_{32} \cdot p_{21} = \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{24}$.

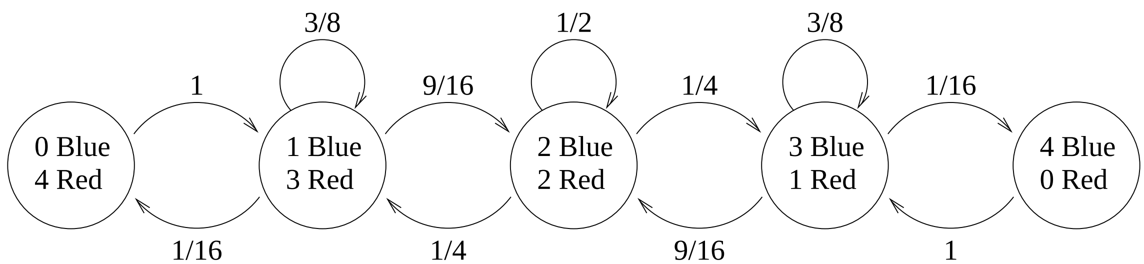
(e) $\mathbf{P}(\{\text{process in } S_3 \text{ immediately after the } N\text{th trial}\})$

$$\begin{aligned}
&= \mathbf{P}(\{\text{moves to } S_3 \text{ in first trial and stays in } S_3 \text{ for next } N-1 \text{ trials}\}) \\
&= \frac{1}{3}\left(\frac{1}{4}\right)^{n-1} \qquad \text{for } n = 1, 2, 3, \ldots
\end{aligned}$$

2. (a) The states represent the various color combinations of the balls in the two urns at a certain stage. Determining the contents of the first urn also determines the contents of

the second urn. All the balls that are not in the first urn must be in the second urn. Therefore we define our states to be the number of blue and red balls in the first urn. State transitions correspond to the ball exchanges between the two urns. At least 5 states are needed to represent all the possible ball combinations in the first urn, since the first urn can contain 0, 1, 2, 3, or 4 blue balls and the remaining balls must all be red.

(b) The corresponding state transition diagram of the balls in the first urn is shown in the figure below:



(c) The minimum number of states is 19. Following the logic in part (a), we only need consider the possible combinations of 2 out of 3 colors. The third color can be duduced from the constraint of having 6 balls in a urn. The possible states for red and white balls in the urn 1 is

$$\begin{aligned}
&\text{Possible combinations of red and white balls in urn 1} = \\
&\qquad \text{0 red + 2-4 white} = \text{3 states} \\
&\qquad \text{1 red + 1-4 white} = \text{4 states} \\
&\qquad \text{2 red + 0-4 white} = \text{5 states} \\
&\qquad \text{3 red + 0-3 white} = \text{4 states} \\
&\qquad \text{4 red + 0-2 white} = \text{3 states} \\
&= \quad \text{19 states}
\end{aligned}$$

3. The sequence of letters recorded by the monitoring device $\boxed{\text{cannot}}$ be modeled as the state history of a Markov process.

Let M_k denote the output of the monitoring device at time k. Note that $M_k = L$ or R. Let D_k denote the tile on which the mouse is at time k.

$$\begin{aligned}
&\mathbf{P}(M_k = L | M_{k-1} = R, M_{k-2} = L) \\
&\quad = \quad \mathbf{P}(D_k = T_{n/2} | D_{k-1} = T_{n/2+1}, D_{k-2} = T_{n/2}) \\
&\quad = \quad \mathbf{P}(D_k = T_{n/2} | D_{k-1} = T_{n/2+1}) \\
&\quad = \quad 1/2 \\
&\mathbf{P}(M_k = L | M_{k-1} = R, M_{k-2} = R, M_{k-3} = L) \\
&\quad = \quad \mathbf{P}(D_k = T_{n/2} | D_{k-1} = T_{n/2+2}, D_{k-2} = T_{n/2+1}, D_{k-3} = T_{n/2}) \\
&\quad = \quad \mathbf{P}(D_k = T_{n/2} | D_{k-1} = T_{n/2+2}) \\
&\quad = \quad 0
\end{aligned}$$

Since $\mathbf{P}(M_k = L|M_{k-1} = R, M_{k-2} = R, M_{k-3} = L)) \neq \mathbf{P}(M_k = L|M_{k-1} = R, M_{k-2} = L)$, it is clear that knowing the immediate previous state, the probability of the current state still depends on the knowledge of other previous states. Therefore, the sequence of letters recorded by the monitoring device does not satisfy the Markov property.