6.431 Fall 2004 Final
Tuesday 14th of December, 1:30-4:30pm
DO NOT TURN THIS QUIZ OVER UNTIL YOU ARE TOLD TO DO SO

Your Full Name:	
Recitation Instructor and TA's Names:	

- This quiz has 4 problems, over 17 pages, worth 300 points. Parts are not necessarily in order of difficulty.
- Write your solutions in the space provided in this quiz itself. We will not consider any work elsewhere. Blue books are provided for your scratch work. Hand in both this quiz and the blue books.
- A correct answer does not guarantee full credit and a wrong answer does not guarantee loss of credit. You should concisely indicate your reasoning and show all relevant work. The grade on each problem is based on our judgment of your level of understanding as reflected by what you have written.
- This is a closed-book exam except for three double-sided, 8.5 by 11 formula sheets.

Problem		**Your Score**	**Problem**		**Your Score**
1.1	**20**				
1.2	**10**		**1.3**	**10**	
1.4	**15**		**1.5**	**30**	
1.6	**35**		**2.1**	**10**	
2.2	**20**		**2.3**	**10**	
2.4	**10**		**2.5**	**15**	
2.6	**15**		**2.7**	**25**	
3	**39**		**4.1**	**12**	
4.2	**12**		**4.3**	**12**	
			Total (300 points)		

Problem 1

Regular emails arrive according to a Poisson process with rate $\lambda_r = 2$ messages per hour. Spam email arrives according to a Poisson process with rate $\lambda_s = 8$ messages per hour, independently of the regular emails. Each regular, non-spam, email is an invitation to a party with probability $p = 0.05$, independently of everything else.

1. (20 points) It takes you 2 seconds to recognize and delete a spam message. The time it takes you to read and answer regular emails is uniformly distributed between 60 and 120 seconds. Find the expectation and the variance of the time it will take you to deal with all the email you receive from noon to 10pm.

2. (10 points) You just got a new email. What is the probability that it is a party invitation?

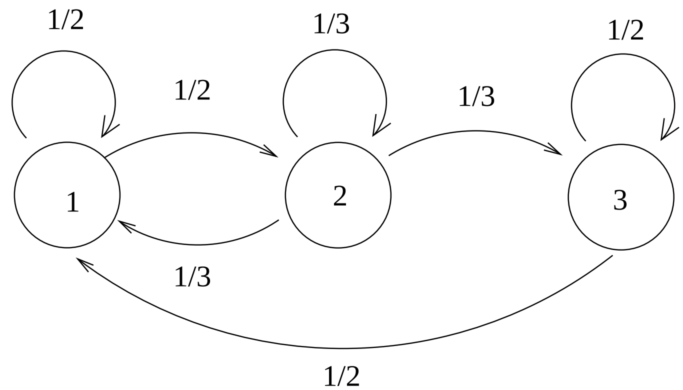
3. (10 points) Find the PMF of the number of party invitations you will receive from noon to 10pm.

4. (15 points) Find the PMF of the number of party invitations you will receive from noon to 10pm, given that you got at least one email during that time.

5. (30 points) We are interested in probability that out of the first 100 emails you receive, exactly 80 are spam. Write down the exact expression for the probability and find a *good* approximation of it (we are looking for a number; writing down a formula is insufficient).

6. (35 points) Right after you checked your email, you fell asleep for a random amount of time, described by an exponential PDF with parameter 1. You wake up and find that three regular emails have arrived. Find the conditional PDF of your sleeping time given this information.

Problem 2



Let X_n be a Markov chain with transition probabilities as shown in the figure. The chain starts in state 1 with probability 1/3, and in state 2 with probability 2/3.

1. (10 points) Identify recurrent states and transient states. Identify recurrent classes.

2. (20 points) Find the steady-state probabilities π_i.

3. (10 points) Find the expected time until we enter state 3 for the first time.

Let Y_n be the time that the third state is visited for the the n^{th} time, and let $T_n = Y_{n+1} - Y_n$ be the time between consecutive visits.

4. (10 points) Find the expectation of T_n.

5. (15 points) Does the sequence T_n converge in probability? If yes, to what value? If no, briefly explain why.

6. (15 points) Does the sequence $Q_n = \frac{1}{n} \sum_{i=1}^{n} T_i$ converge in probability? If yes, to what value? If no, briefly explain why.

At the beginning of the process, the initial state X_0 (1 or 2) is transmitted to a remote site. The received signal is $Y = X_0 + W$, where W is an independent noise term, distributed according to the standard normal distribution (i.e., its mean is 0 and its variance is 1).

7. (25 points) Find the conditional PDF of Y^2 given that $X_0 = 1$.

Problem 3

(39 points) Let X_1 and X_2 be independent, exponentially distributed random variables with mean π. Find the distribution of $\frac{X_1}{X_1+X_2}$. (Note: assume 0/0 is 0).

Problem 4

In the following problems, please indicate "True" or "False", with your justification.

1. (12 points) If the sequence X_n converges in probability to some c, then c is the limit of the sequence of means, $E[X_n]$.

2. (12 points) Let X, Y be independent exponentially distributed random variables with parameters λ, μ, respectively. Then $\min(X, Y)$ is an exponentially distributed random variable with parameter $\min(\lambda, \mu)$.

3. (12 points) Let X be a zero-mean Gaussian and let Y be a Bernoulli random variable that takes values -1 and 1 with equal probability. X and XY are Gaussian and uncorrelated, but not independent.