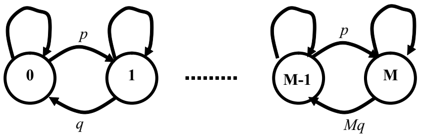
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2005)

Problem Set 10 Solutions
Due: May 4, 2005

1. The outcome of the next game depends on the outcome of the past two games, thus we need a 4 state Markov chain to model the process. The states will be all the ordered pairs of outcomes of the past two games, where the second entry marks the outcome of the most recent game:
$$\{S_1 = (W, W); S_2 = (W, L); S_3 = (L, W); S_4 = (L, L)\}$$
Therefore the transition probability matrix will be:
$$[P] = \begin{pmatrix} .7 & .3 & 0 & 0 \\ 0 & 0 & .4 & .6 \\ .5 & .5 & 0 & 0 \\ 0 & 0 & 0.2 & .8 \end{pmatrix}$$
We see from the chain that it has one recurrent, aperiodic class, and therefore is ergodic. Thus we can apply the fundamental theorem. The theorem tells us that if we can find positive numbers $\{\pi_i\}$ that satisfy:
$$\pi_j = \sum_i \pi_i p_{ij}, \text{ and } \sum_i \pi_i = 1$$
then these $\{\pi_i\}$ are in fact the steady state probabilities. Solving the linear system, we find that:
$$\pi_1 = \frac{5}{20}, \pi_2 = \frac{3}{20}, \pi_3 = \frac{3}{20}, \pi_4 = \frac{9}{20}$$
and the desired probability is $\pi_1 + \pi_3 = \frac{8}{20} = \frac{2}{5}$.

The long run probability that the team will win its next game, i.e., the sum probability of states (W,W) and (L,W), is $\frac{2}{5}$. This conclusion can be reached via $\mathbf{P}(\text{winning next game}) = 0.7 * \pi_1 + 0.4 * \pi_2 + 0.5 * \pi_3 + 0.2 * \pi_4$.

2. (a) We define a Markov chain with states $0, 1, \cdots, M$, corresponding to the number of customers in the house. Assume that $Mq < 1$, the transition probability graph is given as follows,



   (b) For the above Markov Chain, the local balance equations are
$$\pi_i p = \pi_{i+1}(i + 1)q, \quad i = 0, 1, \cdots, M - 1.$$
We define $\rho = p/q$, and obtain $\pi_{i+1} = \frac{\rho}{i+1}\pi_i$, which leads to
$$\pi_i = \frac{\rho^i}{i!}\pi_0, \quad i = 0, 1, \cdots, M - 1.$$

By using the normalization equation, $1 = \pi_0 + \pi_1 + \cdots + \pi_M$, we obtain

$$1 = \pi_0(1 + \frac{\rho^1}{1!} + \frac{\rho^2}{2!} + \cdots + \frac{\rho^M}{M!}),$$

and

$$\pi_0 = \frac{1}{\sum_{k=0}^{M} \frac{\rho^k}{k!}}.$$

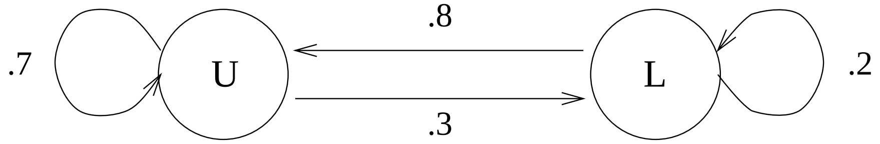
Using the equation $\pi_i = \frac{\rho^i}{i!}\pi_0$, the steady-state probabilities are

$$\pi_i = \frac{\frac{\rho^i}{i!}}{\sum_{k=0}^{M} \frac{\rho^k}{k!}}.$$

Therefore, the average number of customers in the house is given by

$$\bar{N} = \sum_{i=0}^{M} i\pi_i = \rho * \frac{\sum_{i=0}^{M-1} \frac{\rho^i}{i!}}{\sum_{k=0}^{M} \frac{\rho^k}{k!}}.$$

3. The state-transition diagram is the following:



(a) We are interested in finding the steady-state probabilites of the states in this Markov chain. Since this is a birth-death process, we use the local balance equations based on the frequency of transitions between two successive states and the normalization equation to solve for π_U and π_L.

$$\begin{aligned} \pi_L &= \frac{\pi_U \cdot 3/10}{8/10} = \frac{3}{8}\pi_U \\ 1 &= \pi_L + \pi_U. \end{aligned}$$

Solving this system of equations, we get,

$$\pi_U = \frac{8}{11} \qquad \pi_L = \frac{3}{11}.$$

Thus,

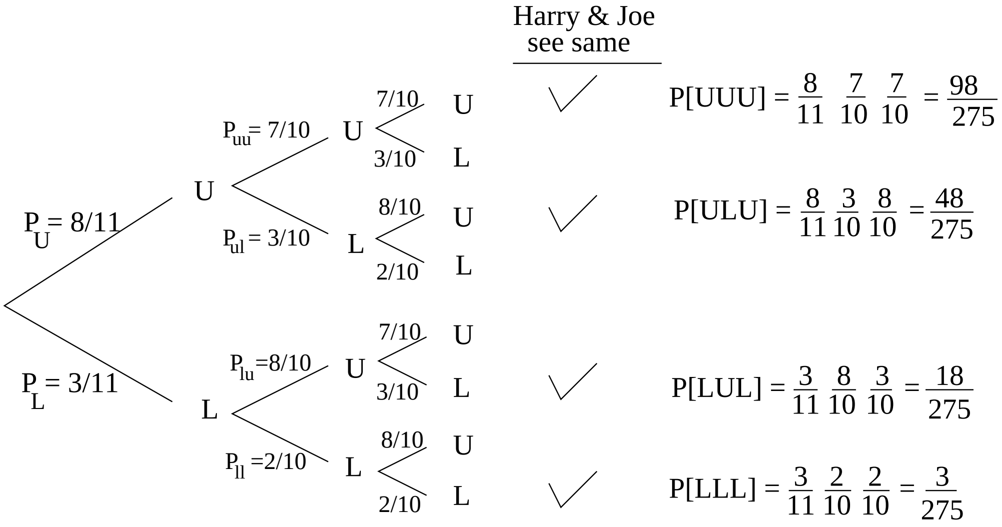
$$\mathbf{P}(\text{he unlocks the door}) = \pi_L \cdot p_{LU} = \frac{3}{11} \cdot \frac{8}{10} = \frac{12}{55}$$

and

$$\mathbf{P}(\text{he locks the door}) = \pi_U \cdot p_{UL} = \frac{8}{11} \cdot \frac{3}{10} = \frac{12}{55}.$$

So, the two events are equally likely.

(b) We can draw a tree of the possible outcomes of Mean Variance's two visits between Joe's arrival and Harry's.



$$\mathbf{P}(\text{both Joe and Harry see the same condition}) = \frac{98}{275} + \frac{48}{275} + \frac{18}{275} + \frac{3}{275} = \boxed{\frac{167}{275}}.$$

(c) Define

X = number of visits from hiring to locking

Y = number of visits from locking to unlocking.

W = number of visits from hiring to unlocking (this is the random variable of interest)

Note that $W = X + Y$. X is a geometric random variable with success probability equal to 0.3 and Y is a geometric random variable with success probability equal to 0.8:

$$\begin{aligned} p_X(x) &= \frac{3}{10}\left(\frac{7}{10}\right)^{x-1}, x = 1, 2, 3, \ldots \\ p_Y(y) &= \frac{8}{10}\left(\frac{2}{10}\right)^{y-1}, y = 1, 2, 3, \ldots \end{aligned}$$

Using the linearity property of expectation and the expected value of a geometric random variable we obtain,

$$\begin{aligned} E[W] &= E[X] + E[Y] \\ &= \frac{10}{3} + \frac{10}{8} \\ &\approx 4.583 \end{aligned}$$

(d) We define a state of two random variables (S, I), where $S \in U, L$ denotes whether the door is unlocked or locked, and $I \in 1, 0$ denotes whether he visits the door at the begining of the hour. Therefore, there are four states in total:

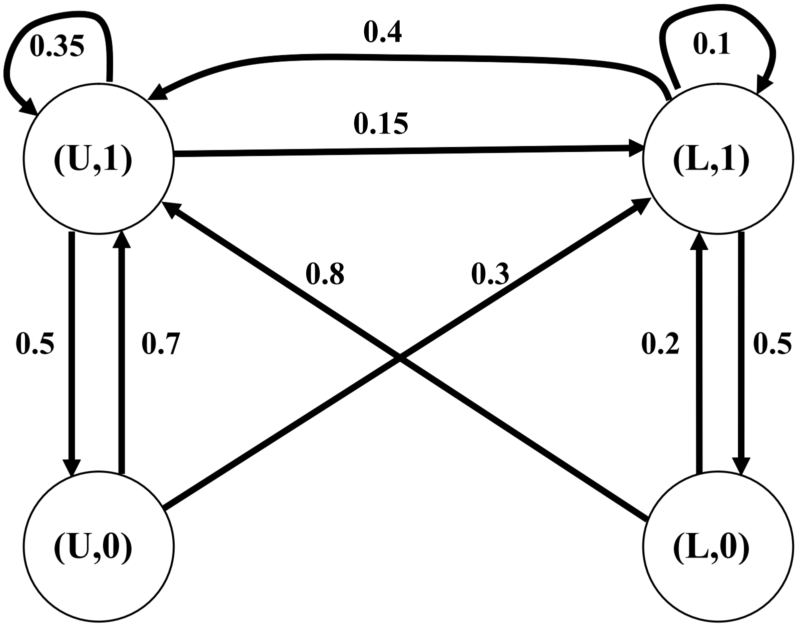
$(U,1)$ = Door is unlocked and he visits the door at the beginning of the hour

$(U,0)$ = Door is unlocked and he does not visit the door at the beginning of the hour

$(L,1)$ = Door is locked and he visits the door at the beginning of the hour

$(L,0)$ = Door is locked and he does not visit the door at the beginning of the hour

Using the above states, the transition probability graph is given by



For the above Markov chain, the steady state probabilites satisfy the following equations

$$\begin{aligned}
\pi_{U,1} &= 0.35\pi_{U,1} + 0.7\pi_{U,0} + 0.4\pi_{L,1} + 0.8\pi_{L,0} \\
\pi_{U,0} &= 0.5\pi_{U,1} \\
\pi_{L,1} &= 0.15\pi_{U,1} + 0.3\pi_{U,0} + 0.1\pi_{L,1} + 0.2\pi_{L,0} \\
\pi_{L,0} &= 0.5\pi_{L,1}
\end{aligned}$$

Moreover, we have the following normalization equation,

$$\pi_{U,1} + \pi_{U,0} + \pi_{L,1} + \pi_{L,0} = 1.$$

Solving these equations, we obtain

$$\begin{aligned}
\pi_{U,1} &= \frac{16}{33} \\
\pi_{U,0} &= \frac{8}{33} \\
\pi_{L,1} &= \frac{6}{33} \\
\pi_{L,0} &= \frac{3}{33}
\end{aligned}$$

Therefore, the probability of seeing the door unlocked is

$$\mathbf{P}(U) = \pi_{U,1} + \pi_{U,0} = \frac{16}{33} + \frac{8}{33} = \frac{8}{11}.$$

Comparing to Part (a), the answer does not change at all.

(e) As in Part (b), we can draw a tree of the possible states between the two visits. Here, we list all the possible state transitions which satify the requirement that both the door states are the same for the two visits, as follows,

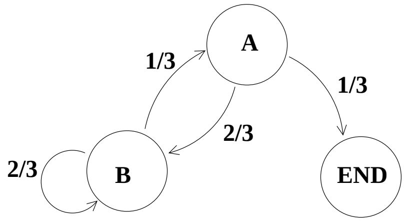
$(U,1)- > (U,1)- > (U,1)$ with probability of $\frac{16}{33} \times \frac{35}{100} \times \frac{35}{100}$
$(U,1)- > (U,1)- > (U,0)$ with probability of $\frac{16}{33} \times \frac{35}{100} \times \frac{50}{100}$
$(U,1)- > (U,0)- > (U,1)$ with probability of $\frac{16}{33} \times \frac{50}{100} \times \frac{70}{100}$
$(U,1)- > (L,1)- > (U,1)$ with probability of $\frac{16}{33} \times \frac{15}{100} \times \frac{40}{100}$
$(U,0)- > (U,1)- > (U,1)$ with probability of $\frac{8}{33} \times \frac{70}{100} \times \frac{35}{100}$
$(U,0)- > (U,1)- > (U,0)$ with probability of $\frac{8}{33} \times \frac{70}{100} \times \frac{50}{100}$
$(U,0)- > (L,1)- > (U,1)$ with probability of $\frac{8}{33} \times \frac{30}{100} \times \frac{40}{100}$
$(L,1)- > (U,1)- > (L,1)$ with probability of $\frac{6}{33} \times \frac{40}{100} \times \frac{50}{100}$
$(L,1)- > (L,1)- > (L,1)$ with probability of $\frac{6}{33} \times \frac{10}{100} \times \frac{10}{100}$
$(L,1)- > (L,1)- > (L,0)$ with probability of $\frac{6}{33} \times \frac{10}{100} \times \frac{50}{100}$
$(L,1)- > (L,0)- > (L,1)$ with probability of $\frac{6}{33} \times \frac{50}{100} \times \frac{20}{100}$
$(L,0)- > (U,1)- > (L,1)$ with probability of $\frac{3}{33} \times \frac{80}{100} \times \frac{50}{100}$
$(L,0)- > (L,1)- > (L,1)$ with probability of $\frac{3}{33} \times \frac{20}{100} \times \frac{10}{100}$
$(L,0)- > (L,1)- > (L,0)$ with probability of $\frac{3}{33} \times \frac{20}{100} \times \frac{50}{100}$

The probability with which Joe and Harry see the same condition is equal to the sum of all the probabilites of the above transitions. It follows that

$$\mathbf{P}(\text{both Joe and Harry see the same condition}) = \frac{173}{275},$$

which is larger than the answer in Part (b).

4. We set up the Markov chain shown below.



States A and B indicate the type of the most recent event, except if a second A in a row occurs, in which case we move to state END. The problem is to find the expected time until we enter state END, starting from state B. Note that the times between transitions are i.i.d. exponential, with mean 1/3. This comes from the fact that the arrivals of type A and B can be seen as arrivals of a single merged Poisson process with rate 3. Also, notice that even though the last two arrivals before absorption into END are going to be A, the expected length of each of the last two interarrival times is still going to be $\frac{1}{3}$, and not 1 as one might mistakenly assume. In general, in a merged Poisson process, given that the next arrival is going to be of a particular type (say type A), the expected time until the next arrival is still just the expected interarrival time of the merged process(not the expected interarrival time of arrivals of type A). Thus, the desired expected time is 1/3 times the expected number of transitions. (We are using here the formula for the expectation of a sum of a random number of i.i.d. random variables.) Let t_i be the expected number of transitions until the end starting from state i. We have

$$t_B = 1 + \frac{2}{3}t_B + \frac{1}{3}t_A,$$

$$t_A = 1 + \frac{2}{3}t_B.$$

We solve and find $t_B = 12$, $t_A = 9$. Thus, the desired expected time is $12/3 = 4$.