Problem Set 1: Alternative solution to G1
Due: February 9, 2005

G1. A more general version of this problem is the following. Let $A_1, A_2, \ldots$ be a countably infinite sequence of events such that $A_{n+1} \subset A_n$ for all $n = 1, 2, \ldots$. Let $A = \bigcap_n A_n$. [1] Show that $\lim_n \mathbf{P}(A_n) = \mathbf{P}(A)$.

Solution. First note that for all n

$$\begin{aligned} \mathbf{P}(A_n) &= \mathbf{P}((A_n \cap A^c) \cup (A_n \cap A)) \\ &= \mathbf{P}(A_n \cap A^c) + \mathbf{P}(A_n \cap A) && \text{[additivity]} \\ &= \mathbf{P}(A_n \cap A^c) + \mathbf{P}(A). \end{aligned}$$

Then note that similarly for all n

$$\begin{aligned} \mathbf{P}(A_1 \cap A^c) &= \mathbf{P}\left(\bigcup_{i=1}^{n-1} (A_i \cap A_{i+1}^c) \cup (A_n \cap A^c)\right) \\ &= \sum_{i=1}^{n-1} \mathbf{P}(A_i \cap A_{i+1}^c) + \mathbf{P}(A_n \cap A^c). && \text{[additivity]} \end{aligned}$$

Taking the limit at both sides, we have [2]

$$\begin{aligned} \mathbf{P}(A_1 \cap A^c) &= \sum_{i=1}^{\infty} \mathbf{P}(A_i \cap A_{i+1}^c) + \lim_n \mathbf{P}(A_n \cap A^c) \\ &= \mathbf{P}\left(\bigcup_i (A_i \cap A_{i+1}^c)\right) + \lim_n \mathbf{P}(A_n \cap A^c) && \text{[additivity]} \\ &= \mathbf{P}(A_1 \cap A^c) + \lim_n \mathbf{P}(A_n \cap A^c). \end{aligned}$$

Thus, we obtain

$$\lim_n \mathbf{P}(A_n \cap A^c) = 0,$$

which gives the result

$$\lim_n \mathbf{P}(A_n) = \lim_n \mathbf{P}(A_n \cap A^c) + \mathbf{P}(A) = \mathbf{P}(A).$$

This completes the proof.

[1] This is also often denoted by $A_n \searrow A$ where we say that A_n *decreases* to A.
[2] Proving that $A_1 \cap A^c = \bigcup_i (A_i \cap A_{i+1}^c)$ in the last step is easy.