Problem Set 3 Solutions
Due: February 23, 2005

1. To find the probability, we will find the number of favorable outcomes, and divide by the total number of possible outcomes. There are $\binom{10}{8}$ favorable outcomes, i.e. successful combinations that will open the lock. There are $\binom{90}{8}$ total number of ways to choose 8 numbers out of 90, and therefore the probability that the burglar will open the vault on his first try is:

$$\frac{\binom{10}{8}}{\binom{90}{8}}.$$

2. We know that the probability of the whole sample space equals to one.

$$\mathbf{P}(\text{non-compact}) + \mathbf{P}(\text{compact}) = 1.$$

And we also know that $\mathbf{P}(\text{non-compact}) = 2\mathbf{P}(\text{compact})$. So,

$$\mathbf{P}(\text{non-compact}) = \frac{2}{3} \qquad \mathbf{P}(\text{compact}) = \frac{1}{3}.$$

(a) Since the fact that the last car is non-Compact does not tell anything about the made of the next car, these two events are independent. Therefore,

$$\mathbf{P}(\text{next is non-compact} \mid \text{last car non-compact}) = \mathbf{P}(\text{next is non-compact}) = \frac{2}{3}$$

Also, the next k cars are also independent of the last one we saw. Then, the probability that we have to wait k more cars to see another non-Compact given that the last car we saw is non-Compact is simply the probability that the next $k-1$ cars are all Compacts and the k^{th} car is non-Compact. And this probability equals to

$$\left(\frac{1}{3}\right)^{k-1}\left(\frac{2}{3}\right)$$

(b) i. The probability that the i^{th} car is a Compact is $1/3$. The probability that the rest are non-Compact is $(2/3)^9$. Futhermore, there are ten choices for i, i.e. the 1^{st} to the 10^{th}. So, the probability of this event is

$$10\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^9.$$

ii. If the 10^{th} car was the 4^{th} Compact, that means 3 of the first 9 cars must be Compacts. So the number of distinct sequences is $\binom{9}{3}$.

In each sequence, 4 Compacts and 6 non-Compacts are obrseved. So, the probability of each sequence is

$$\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^6.$$

iii. If the 10^{th} car was the k^{th} Compact, that means $k-1$ of the first 9 cars must be Compacts and the rest of the first 9 cars must be non-Compacts. Each sequence has the probability of $(\frac{1}{3})^k(\frac{2}{3})^{10-k}$. And the number of ways we can arrange the $k-1$ Compacts in the sequence of 9 cars is $\binom{9}{k-1}$. So the probability for this event is

$$\binom{9}{k-1}\left(\frac{1}{3}\right)^k\left(\frac{2}{3}\right)^{10-k}.$$

(c) i. Each car can be either Compact or non-Compact independent of other cars. Therefore, the number of all possible distinct car sequences is 2^{25}.

ii. These sequences can either end when the number of Compacts reaches 13 or the number of non-Compacts does. Suppose the number of Compacts reaches 13 first and this happens at the $x+1^{st}$ car for $12 \le x \le 24$. Among the first x cars, there must be 12 Compactss and $x-12$ non-Compacts. Summing over all possible x, and double the result to properly account for the sequences where the number of non-Compacts reaches 13 first. The total number of distinct sequences is

$$2\sum_{x=12}^{24}\binom{x}{12} = 10400600$$

Alternatively, the problem can be approached in a different way. Again, assume that the number of Compacts reaches 13 first. Now choose 13 cars out of the 25 cars to be Compacts. The length of the sequence (as described as $(x+1)$ above) is actually determined by how we pick the 13 cars. For example, if we pick the first 13 cars to be all Compactd, x will be automatically set as 12. Therefore, the total number of ways to arrange the 13 Compacts is $\binom{25}{13}$, i.e. total number of distinct sequences is

$$2\binom{25}{13} = 10400600$$

3. (a) We recognize that A is a binomial random variable (sum of n independent Bernoulli random variables).

$$p_A(a) = \begin{cases} \binom{n}{a}p^a(1-p)^{n-a} & a = 0, 1, \ldots, n \\ 0 & \text{otherwise.} \end{cases}$$

So, we have $\mathbf{E}[A] = np$, $\text{var}(A) = np(1-p)$.

(b) Let L denote the desired event, i.e. L: the first file saved ends up being the only one on its drive. Let FA: the first file is saved on drive A. Let FB: the first file is saved on drive B. We apply the law of total probability and obtain

$$\begin{aligned}\mathbf{P}(L) &= \mathbf{P}(L|FA)\cdot\mathbf{P}(FA) + \mathbf{P}(L|FB)\cdot\mathbf{P}(FB) \\ &= p\cdot(1-p)^{n-1} + (1-p)\cdot p^{n-1}\end{aligned}$$

(c) Let EA: drive A ends up with exactly one file. Let EB: drive B ends up with exactly one file. Finally, let E denote the desired event - at least one drive ends up with a total of exactly one file. Notice that $E = EA \cup EB$. In general, we know that, $\mathbf{P}(E) = \mathbf{P}(EA) + \mathbf{P}(EB) - \mathbf{P}(EA \cap EB)$. Now when $n \neq 2$, $\mathbf{P}(EA \cap EB) = 0$. We distinguish two cases:

case (1): $n \neq 2$. We have $\mathbf{P}(E) = \mathbf{P}(EA) + \mathbf{P}(EB) = \binom{n}{1}p(1-p)^{n-1} + \binom{n}{1}(1-p)p^{n-1}$, since both A and B are binomial random variables (with different "success" probabilities).

case (2): $n = 2$. We have $\mathbf{P}(E) = \mathbf{P}(EA) + \mathbf{P}(EB) - \mathbf{P}(EA \cap EB) = 2p(1-p) + 2(1-p)p - 2p(1-p) = 2p(1-p)$.

(d) By linearity of expectation, we have $\mathbf{E}[D] = \mathbf{E}[A] - \mathbf{E}[B]$. Both A and B are binomial, and we have $\mathbf{E}[D] = np - n(1-p) = n(2p-1)$. Now since A and B are not independent (i.e. A and B are dependent) we cannot simply add their variances. Notice however that $A+B = n$, hence $B = n-A$. Therefore, $\text{var}(D) = \text{var}(A-B) = \text{var}(A-n+A) = \text{var}(2A-n) = 4\text{var}(A) = 4np(1-p)$.

(e) Let C: both of the first two files to be saved go onto drive A. Consider the Bernoulli random variable X_i with parameter $p, i = 3, \ldots, n$. Notice that $\mathbf{E}[A|C] = \mathbf{E}[2 + \sum_{i=3}^{n} X_i] = 2 + \mathbf{E}[\sum_{i=3}^{n}] = 2 + (n-2)p$.
This is because each of the remaining $n-2$ trials still have success probability p. Now $\text{var}(A|C) = \text{var}(2 + \sum_{i=3}^{n} X_i) = \text{var}(\sum_{i=3}^{n} X_i) = \sum_{i=3}^{n}(X_i)$ (by independence). So, $\text{var}(A|C) = (n-2)p(1-p)$. Finally, we desire $p_{A|C}(a)$.
Notice that in our new universe $A = 2 + \sum_{i=3}^{n} X_i$. The number of successes in the remaining trials is binomial. We just shift the PMF so that, for instance, $p_{A|C}(2) = \mathbf{P}(0 \text{ successes in } n-2 \text{ trials})$. So we have

$$p_{A|C}(a) = \begin{cases} \binom{n-2}{a-2}p^{a-2}(1-p)^{n-a} & a = 2, 3, \ldots, n \\ 0 & \text{otherwise.} \end{cases}$$

4. Let the event A be the event that the professor teaches her class. Since the events B and B^c are mutually exclusive and collectivley exhaustive, we can use the identity:

$$\mathbf{P}(A) = \mathbf{P}(A \cap B) + \mathbf{P}(A \cap B^c)$$

where $\mathbf{P}(A \cap B) = \mathbf{P}(A|B) \cdot \mathbf{P}(B)$. Given the weather on a certain day, the probability that the professor will teach her class is the complement of the binomial distribution function:

$$\mathbf{P}(\text{Prof. will teach}|\text{The weather}) = 1 - \sum_{i=0}^{k-1} \binom{n}{i} \cdot p^i (1-p)^{n-i}$$

where $p = p_b, p_g$ depending on the day. Therefore,

$$\mathbf{P}(\text{Prof. will teach}) = \mathbf{P}(B) \cdot (1 - \sum_{i=0}^{k-1} \binom{n}{i} \cdot p_b^i (1-p_b)^{n-i}) + (1 - \mathbf{P}(B)) \cdot (1 - \sum_{i=0}^{k-1} \binom{n}{i} \cdot p_g^i (1-p_g)^{n-i}).$$

5. Suppose the president decides to investigate A first. Then her expected costs will be:

$$\mathbf{E}[\text{costs}] = D_A + pR_A + (1-p) \cdot (D_B + R_B)$$

where as if she investigates B first, then

$$\mathbf{E}[\text{costs}] = pD_A + pR_A + D_B + (1-p) \cdot R_B$$

In order that the first be smaller than the second, we need:

$$pD_B > (1-p)D_A$$