Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2005)

Problem Set 1
Due: February 9, 2005

1. Each of two people choose at random a number x and y, respectively, between zero and one. We take "at random" to mean "according to the uniform probability law" introduced in lecture. Consider the following events:

$$\begin{aligned} A &= \{\text{The magnitude of the difference of the two numbers is at most } 1/3.\} \\ B &= \{\text{None of the numbers exceeds } 2/3.\} \\ C &= \{\text{The two numbers are equal.}\} \end{aligned}$$

Find the following probabilities: $P(A)$, $P(B)$, $P(A \cap B)$, $P(C)$.

2. Twenty distinct cars park in the same parking lot everyday. Ten of these cars are US-made, while the other ten are foreign-made. This parking lot has exactly twenty spaces, and all are in a row, so the cars park side by side each day. The drivers have different schedules on any given day, however, so the position any car might take on a certain day is random.

   (a) In how many different ways can the cars line up?

   (b) What is the probability that on a given day, the cars will park in such a way that they alternate (e.g., US-made, foreign-made, US-made, foreign-made, etc)?

3. Express each of the following events in terms of the events A, B and C as well as the operations of complementation, union and intersection:

   (a) at least one of the events A, B, C occurs;

   (b) at most one of the events A, B, C occurs;

   (c) none of the events A, B, C occurs;

   (d) all three events A, B, C occur;

   (e) exactly one of the events A, B, C occurs;

   (f) events A and B occur, but not C;

   (g) either event A occurs or, if not, then B also does not occur.

In each case draw the corresponding Venn diagrams.

4. Let A and B be two events. Show the following inequalities. Under what conditions on A and B are these inequalities satisfied as equations? A mathematical derivation is required, but you can use Venn diagrams to guide your thinking.

   (a) $P(A) + P(B) \geq P(A \cup B) \geq \max\{P(A), P(B)\}$.

   (b) $\min\{P(A), P(B)\} \geq P(A \cap B) \geq P(A) + P(B) - 1$.

5. Bob has a peculiar pair of four-sided dice. When he rolls the dice, the probability of any particular outcome is proportional to the product of the outcome of each die. All outcomes that result in a particular product are equally likely.

   (a) What is the probability of the product being even?

(b) What is the probability of Bob rolling a 2 and a 3?

G1†. Consider an experiment whose sample space is the real line. Let $\{a_n\}$ an increasing sequence of numbers that converges to a and $\{b_n\}$ a decreasing sequence that converges to b. Show that

$$\lim_{n\to\infty} \mathbf{P}([a_n, b_n]) = \mathbf{P}([a, b]).$$

Here, the notation $[a, b]$ stands for the closed interval $\{x \mid a \leq x \leq b\}$. *Note:* This result seems intuitively obvious. The issue is to derive it using the axioms of probability theory.

†Required for 6.431; optional for 6.041