6.431 Spring 2005 Final Exam
Tuesday, May 17, 1:30–4:30 p.m.

DO NOT TURN THIS EXAM OVER UNTIL YOU ARE TOLD TO DO SO

- You have 3 hours to complete the exam.
- Write your solutions in the exam booklet. We will not consider any work not in the exam booklet.
- This exam has three problems, each with multiple parts, that are not necessarily in order of difficulty.
- You may give an answer in the form of an arithmetic expression (sums, products, ratios, factorials) of numbers that could be evaluated using a calculator. Expressions like $\binom{8}{3}$ or $\sum_{k=0}^{5}(1/2)^k$ are also fine.
- A correct answer does not guarantee full credit and a wrong answer does not guarantee loss of credit. You should concisely indicate your reasoning and show all relevant work. The grade on each problem is based on our judgment of your level of understanding as reflected by what you have written.
- This is a closed-book exam except for three handwritten, 2-sided 8.5x11 formula sheets plus a calculator.
- Be neat! If we can't read it, we can't grade it.
- At the end of the exam, turn in your solutions along with this exam (this piece of paper).

Write your name, your recitation instructor's name, and your TA's name on the front of the booklet. (3 points)

Problem 1: (26 points) Three friends, Xena, Yvonne, and Zelda, decide to run the Boston marathon. For each of them the time required to complete the marathon is a continuous random variable uniformly distributed between 4 hours and 6 hours. The running times of all contestants are independent.

(a) (4 points) Find the probability density function of the race time of the winner.

(b) (4 points) Find the probability density function of the race time of the contestant that arrives in the last place.

(c) (6 points) Find the probability density function of the race time of the contestant that arrives in second place. What is the expected value of this arrival time? Explain.

(d) (4 points) Does the answer to the previous question change if the running times are dependent? Prove or disprove.

After the marathon, a one-hour long TV show interviews the three friends, with 1/2 of the time devoted to the winner, and 1/3 and 1/6 to the second and third, respectively.

You are at home, and you do not know the results of the marathon. You want to find out who won, so you turn on the TV at a random time during the show. You see Zelda on the TV screen.

(e) (4 points) What's the probability that Zelda won the race? What's the probability density function of Zelda's race time?

(f) (4 points) Does the probability of Zelda being the winner change, if you observe her talking non-stop for five minutes? How does the probability distribution of her running time change?

Note: If two or more contestants arrive at exactly the same time, the ranking is decided by random tie-breaking, with equal probabilities.

Problem 2. (36 points) Two kinds of vehicles go to Harvard Square from 77 Mass. Ave.: taxis and buses. The interarrival time of taxis, in minutes, is an independent exponential random variable with parameter λ_1, i.e. its PDF is $f_{I_T}(t) = \lambda_1 e^{-\lambda_1 t}$ for $t \geq 0$, while the interarrival time of buses, in minutes, is an independent exponential random variable with parameter λ_2, i.e., its PDF is $f_{I_B}(t) = \lambda_2 e^{-\lambda_2 t}$ for $t \geq 0$.

Suppose Joe and Harry arrive at 77 Mass. Ave. at 7:00 a.m.

(a) (5 points) What is the average time before they see the first vehicle?

(b) (5 points) What is the probability that the first vehicle they see is a bus, and what is the probability that the first vehicle they see is a taxi?

In a taxi, the travel time to Harvard Square, in minutes, is an independent exponential random variable with parameter μ_1, i.e., its PDF is $f_{D_T}(t) = \mu_1 e^{-\mu_1 t}$ for $t \geq 0$. On the other hand, in a bus, the travel time to Harvard Square, in minutes, is an an independent exponential random variable with parameter μ_2, i.e., its PDF is $f_{D_B}(t) = \mu_2 e^{-\mu_2 t}$ for $t \geq 0$.

(c) (7 points) Suppose Joe and Harry arrive at 77 Mass. Ave. at 7:00 a.m., take the first vehicle that passes, and arrive at Harvard Square X minutes later. Find the transform of X.

(d) (7 points) Suppose that a taxi and a bus arrive simultaneously, and Joe takes the taxi while Harry takes the bus. Let Y be the number of minutes from their departure from 77 Mass. Ave. till they meet again at Harvard Square. Find $\mathbf{E}[Y]$.

There are, in fact, two different kinds of buses: fast buses and slow buses. For any bus that arrives at 77 Mass. Ave., it is a fast bus with probability p, and it is a slow bus with probability $1 - p$. Whether a bus is fast or slow is independent of everything else.

(e) (5 points) If they stay at 77 Mass. Ave. for l minutes, how many fast buses will they see on average?

(f) (7 points) If they stay indefinitely, what is the probability that they will see k fast buses before they see k slow buses?

Problem 3. (35 points) A hungry mouse is trapped in a cage with three doors. At each "turn", the mouse gets to open one of the three doors and eat a piece of cheese behind the door if one is present. Each door is chosen with equal probability on each turn, regardless of whether a piece of cheese was found on the previous turn. If no cheese was found on the previous turn, there is a probability of $3/4$ that cheese will be found behind each door on the current turn. If cheese was found on the previous turn and the same door is chosen on the current turn, then there is a probability of 0 that cheese will be found; whilst if cheese was found on the previous turn and a different door is chosen on the current turn, then there is a probability of 1 that cheese will be found.

(a) (7 points) If you observe the mouse's behavior over 1000 turns, in approximately what fraction of turns do you expect the mouse to eat a piece of cheese?

(b) (7 points) Suppose no cheese was found on the previous turn. What is the expected number of turns before the mouse eats a piece of cheese?

(c) (7 points) Suppose no cheese was found on the previous turn. What is the expected number of turns before the mouse eats n pieces of cheese?

(d) (7 points) Suppose cheese was found on the previous turn. Using the Central Limit Theorem, approximate the probability that the number of turns before the mouse eats 100 pieces of cheese exceeds 152.

(e) (7 points) You look into the cage and observe the mouse eating a piece of cheese from behind door number 1. What is the probability that, if you observe the mouse three turns later, it will again be eating a piece of cheese from behind door number 1?