6.041 Spring 2005 Quiz 2
Monday, April 11, 12:05-12:55 p.m.

DO NOT TURN THIS QUIZ OVER UNTIL YOU ARE TOLD TO DO SO

- You have 50 minutes to complete the quiz.
- Write your solutions in the exam booklet. We will not consider any work not in the exam booklet.
- This quiz has two problems that are not necessarily in order of difficulty.
- You may give an answer in the form of an arithmetic expression (sums, products, ratios, factorials) of numbers that could be evaluated using a calculator. Expressions like $\binom{8}{3}$ or $\sum_{k=0}^{5}(1/2)^k$ are also fine.
- A correct answer does not guarantee full credit and a wrong answer does not guarantee loss of credit. You should concisely indicate your reasoning and show all relevant work. The grade on each problem is based on our judgment of your level of understanding as reflected by what you have written.
- This is a closed-book exam except for one double-sided, handwritten, 8.5 by 11 formula sheet plus a calculator.
- Be neat! If we can't read it, we can't grade it.
- At the end of the quiz, turn in your solutions along with this quiz (this piece of paper).

Write your name, your recitation instructor's name, and your TA's name on the front of the booklet. (3 points)

Problem 1: (55 points)

Your friend, Kerry P., took 6.041 last fall and now supplements his income by gambling. When he visits the casino, he plays blackjack for Y hours, where Y is an exponentially-distributed random variable with mean 1. Dealers are changed every hour after he starts play and, for a given dealer, the rate of his earnings (in thousands of dollars per hour) is described by a Gaussian random variable with mean μ and variance σ^2.

Let us denote the integer part of a real number y by $\lfloor y \rfloor$ so, e.g., $\lfloor 2.8 \rfloor = 2$. Then, Kerry P.'s total earnings X is given by

$$X = \sum_{k=1}^{S} Z_k + Z_{S+1} T,$$

where $\{Z_k\}$ is an i.i.d. sequence of Gaussian random variables with mean μ and variance σ^2 (independent of Y), $S = \lfloor Y \rfloor$, and $T = Y - \lfloor Y \rfloor$.

You would like to know whether you should follow Kerry P.'s footsteps, so you conduct a probabilistic analysis of his earnings.

(a) (13 points) Find $\mathbf{E}[X]$.

(b) (14 points) Find $p_S(k)$, the PMF of S, for $k = 0, 1, \ldots$.

(c) (14 points) Find $f_{T|S}(t|k)$, the conditional PDF of T given S, for $k = 0, 1, \ldots$.

You realize that, although it is difficult to obtain a description of the exact distribution of X, you can bound X by

$$U = \sum_{k=1}^{S+1} Z_k$$

and

$$V = \sum_{k=1}^{S} Z_k$$

from above and below, respectively.

(d) (14 points) Find the PDFs or transforms of U and V.

Problem 2: (42 points)

Every night at closing time, Virginie leaves a bar and walks in a straight line down the street such that, at time t after she starts walking, her position X_t has a Gaussian distribution with mean vt and variance $\sigma_X^2 t$. On most nights, Virginie is accompanied by her cat, whose position at time t, Z_t, is such that $Z_t - X_t = Y_t$, where Y_t is a Gaussian-distributed random variable with mean 0 and variance σ_Y^2. The random variables X_t and Y_t are independent.

(a) (14 points) Find the PDF of the cat's position Z_t as a function of t. What is the correlation coefficient between the cat's position and Virginie's position as a function of t?

On one night, at time τ after bar closing time, Virginie's cat is observed at position z.

(b) (14 points) What is the linear least-squares estimate of Virginie's position at time τ?

(c) (14 points) Show that the least-squares estimate of Virginie's position at time τ is the same as the linear least-squares estimate you obtained in part (b).