Tutorial 4 Answers
Week of February 28, 2005

1. The probability that he receives 6 dollars is $\frac{2}{11}$ and the probability that he receives 2 dollars is $\frac{3}{11}$. Therefore his expected gain is:

$$\frac{2}{11} \cdot 6 + \frac{3}{11} \cdot 2 = \frac{18}{11}$$

2. (a)

$$1 = \int_{x=1}^{2} \left(\int_{y=x}^{2} ax \; dy \right) dx \; = \; \int_{x=1}^{2} ax(2-x) \; dx \; = \; a(2^2 - 1^2 - \frac{2^3}{3} + \frac{1^3}{3}) \; = \; \frac{2}{3}a$$

so $a = 3/2$.

(b) For $1 \le y \le 2$,

$$f_Y(y) = \int_1^y ax \; dx \; = \; \frac{a}{2}(y^2 - 1) \; = \; \frac{3}{4}(y^2 - 1)$$

and $f_Y(y) = 0$ otherwise.

(c) First notice that for $1 \le x \le 3/2$,

$$f_{X|Y}(x|3/2) = \frac{f_{X,Y}(x, 3/2)}{f_Y(3/2)} = \frac{ax}{f_Y(3/2)}$$

where

$$f_Y(3/2) = \int_1^{3/2} ax \; dx = \frac{a}{2}((\frac{3}{2})^2 - 1^2) = \frac{5a}{8}$$

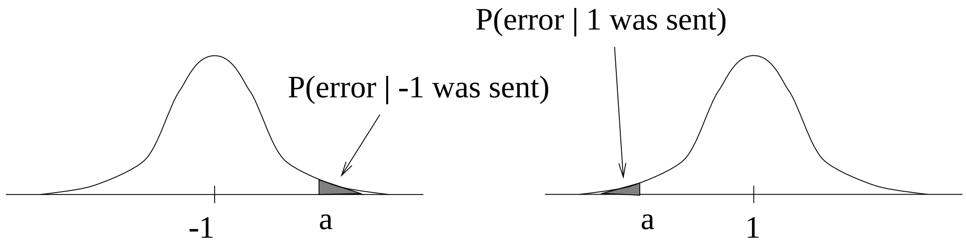
so $f_{X|Y}(x|3/2) = \frac{ax}{\frac{5a}{8}} = \frac{8x}{5}, \quad 1 \le x \le 3/2$. Therefore

$$\mathbf{E}[1/X | Y = 3/2] \; = \; \int_1^{3/2} \frac{1}{x} \frac{8x}{5} \; = \; 4/5.$$

3. (a) Let Z be the random variable representing the additive zero-mean Gaussian noise; that is, $Z \sim N(0, \sigma^2)$. Let S_0 be the event that -1 is sent and S_1 be the event that $+1$ is sent. Let R_0 be the event that we conclude that an encoded signal of -1 was sent based on the received value being less than a. Let R_1 be the event that we conclude that an encoded signal of $+1$ was sent based on the received value being greater than a.

   There are two ways for errors to occur. The true encoded signal could be -1 but we could conclude that the encoded signal of $+1$ was sent. Conditioned on the true encoded signal being -1, the received signal is $Z - 1$; we would erroneously conclude that the encoded signal of $+1$ was sent if $Z - 1 > a$. Similarly, the true encoded signal could be $+1$ but we could conclude that the encoded signal of -1 was sent. In this case, conditioned on the true encoded signal being $+1$, the received signal is $Z + 1$ and we would erroneously conclude that the true signal was -1 if $Z + 1 < a$.

   The figure below illustrates the situations under which errors can occur.



Let Φ such that

$$\Phi(x) = \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{x} e^{-t^2/2} dt.$$

Therefore

$$\begin{aligned}
P(\text{error}) &= P(R_1|S_0)P(S_0) + P(R_0|S_1)P(S_1) \\
&= P(Z - 1 > a)(p) + P(Z + 1 < a)(1 - p) \\
&= p \cdot \left(1 - \Phi\left(\frac{a - (-1)}{\sigma}\right)\right) + (1 - p) \cdot \Phi\left(\frac{a - 1}{\sigma}\right) \\
&= p - p \cdot \Phi\left(\frac{a + 1}{\sigma}\right) + (1 - p) \cdot \left(1 - \Phi\left(\frac{1 - a}{\sigma}\right)\right) \\
&= 1 - p \cdot \Phi\left(\frac{a + 1}{\sigma}\right) - (1 - p) \cdot \Phi\left(\frac{1 - a}{\sigma}\right)
\end{aligned}$$

(b) $P(\text{error}) = 1 - 0.4 \cdot \Phi(\frac{3/2}{1/2}) - 0.6 \cdot \Phi(\frac{1/2}{1/2})$