Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2005)

Recitation 7 Answers
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1. We note that

$$\begin{aligned}\left(\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx\right)^2 &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-x^2/2}dx\int_{-\infty}^{+\infty}e^{-y^2/2}dy\\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}e^{-(x^2+y^2)/2}dx\,dy\\ &= \frac{1}{2\pi}\int_{0}^{2\pi}\int_{0}^{+\infty}e^{-r^2/2}r\,dr\,d\theta\\ &= \int_{0}^{+\infty}e^{-r^2/2}r\,dr\\ &= \int_{0}^{+\infty}e^{-u}du\\ &= -e^{-u}|_0^{+\infty}\\ &= 1,\end{aligned}$$

where for the third equality, we use a transformation into polar coordinates, and for the fifth equality, we use the change of variables $u = r^2/2$. Thus, we have

$$\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx = 1,$$

because the integral is positive. Using the change of variables $u = (x-\mu)/\sigma$, it follows that

$$\int_{-\infty}^{+\infty}f_X(x)dx = \int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/2\sigma^2}dx = \int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-u^2/2}du = 1.$$

2. This probem asks for basic use of the probability density function.

a) The probability that you wait more than 15 minutes is:

$$\int_{15}^{\infty}\frac{1}{15}e^{-\frac{x}{15}}dx = -e^{-\frac{x}{15}}|_{15}^{\infty} = e^{-1}.$$

b) The probability that you wait between 15 and thirty minutes is:

$$\int_{15}^{30}\frac{1}{15}e^{-\frac{x}{15}}dx = -e^{-\frac{x}{15}}|_{15}^{30} = e^{-1} - e^{-2}.$$

3. Given the following events,

A_t: Chip still works at time t.
B: The chip is bad.
G: The chip is good.

We can specify that $\mathbf{P}(A_t|G) = e^{-\alpha t}$ and $\mathbf{P}(A_t|B) = e^{-1000\alpha t}$.

(a) By using the definition of conditional probability and the total probability theorem, we can solve:

$$\mathbf{P}(A_t) = \mathbf{P}(G)\mathbf{P}(A_t|G) + \mathbf{P}(B)\mathbf{P}(A_t|B) = pe^{-\alpha t} + (1-p)e^{-1000\alpha t}$$

(b) By using the definition of conditional probability, we get:

$$\mathbf{P}(B|A_t) = \frac{\mathbf{P}(B \cap A_t)}{\mathbf{P}(A_t)}.$$

Furthermore, $\mathbf{P}(B \cap A_t) = \mathbf{P}(B)\mathbf{P}(A_t|B) = (1-p)e^{-1000\alpha t}$. Therefore,

$$\mathbf{P}(B|A_t) = \frac{(1-p)e^{-1000\alpha t}}{pe^{-\alpha t} + (1-p)e^{-1000\alpha t}}.$$