Problem Set 8
Due: April 20, 2005

1. Suppose your fax machine is transmitting a series of independent codewords. Each codeword consists of three bits, such as "100" or "010". Each bit is independent of others and equally likely to be "1" or "0". The weight of a codeword is found by summing the bits, for example the weight of "111" is 3.

   (a) What is the probability that a codeword with weight at least 2 occurs before a codeword with weight 0?

   (b) Given that a codeword of weight 0 was just sent, what is the probability that the next two codewords also have this weight?

   (c) Whenever "000" or "111" is sent, this will be regarded as a "timing pulse".

      i. Find the PMF of K, the number of codewords up to, but not including, the third timing pulse.

      ii. Find the expectation of L, the number of timing pulses in the first 100 codewords.

      iii. Find the expectation and variance of M, the number of "0"s that have been transmitted *before* the first timing pulse.

      iv. Find the PMF of N, the number of codewords of weight 3 that occur in 100 codewords.

      v. Find the conditional PMF of N given $L = l$.

   (d) My phone is also sending a sequence of codewords, but it employs a different coding scheme. It starts by sending 4-bit codewords. Whenever a "0000" or "1111" occurs, it switches to sending 3-bit codewords, until a "000" or "111" occurs, at which point it switches to 2-bit codewords. When "00" or "11" occurs, the transmission stops.

      i. Find $\mathbf{E}(Q)$, where Q is the number of codewords sent by my phone.

      ii. Find $M_Q(s)$.

2. In a super fast photonic switch that you are designing, each incoming photon is directed to output port "A" with probability p, and to output port B with probability $1-p$, independently. Let R be the total number of photons going to port A and let G be the number of items going to port B in a certain amount of time, during which n photons pass through the switch.

   (a) Determine the PMF, expected value, and variance of the random variable R.

   (b) Evaluate $\mathbf{P}(A)$, the probability that the first photon ends up being the only one sent to its port.

   (c) Find $\mathbf{P}(B)$, the probability that at least one port ends up receiving exactly one photon in this time interval.

   (d) Evaluate the expectation and variance for the difference, $D = R - G$.

   (e) Assume $n \geq 2$. Given that both of the first two input photons go to port A, find the conditional expectation, variance and PMF of R.

3. You are visiting the rainforest, but unfortunately your insect repellent has run out.

(a) As a result, at each second, a mosquito lands on your neck with probability 0.2.

   i. What's the PMF for the time until the first mosquito lands on you?

   ii. What's the expected time until the first mosquito lands on you?

   iii. What if you weren't bitten for the first ten seconds - what would be the expected time until the first mosquito lands on you (from time t=10)?

(b) Instead, imagine the rainforest had only one mosquito, which arrived in the following way: the time of arrival is exponentially distributed with $\lambda = 0.2$.

   i. What's the expected time until the first mosquito lands on you?

   ii. What if you weren't bitten for the first ten seconds - what would be the expected time until the first mosquito lands on you (from time t=10)?

4. Consider the photonic switch of Problem 2, and this time assume that the total number of input photons within a given time interval is Poisson with rate λ. Show that the number of photons received at port A in the same time interval is again Poisson, and has rate $p\lambda$.

5. Based on your understanding of the Poisson process, determine the numerical values of a and b in the following expression and explain your reasoning:

$$\int_t^\infty \frac{\lambda^6 \tau^5 e^{-\lambda\tau}}{5!}\, d\tau = \sum_{k=a}^{b} \frac{(\lambda t)^k e^{-\lambda t}}{k!}.$$