Problem Set 2: Solutions
Due: February 16, 2005

1. a) In order to wind up in the same place after two steps, the tightrope walker can either step forwards, then backwards, or vice versa. Therefore the required probability is:

$$2 \cdot p \cdot (1-p).$$

b) The probability that after three steps he will be one step ahead of his starting point is the probability that out of 3 steps in total, 2 of them are forwards, and one is backwards. This equals:

$$\binom{3}{1} \cdot p^2 \cdot (1-p).$$

c) Given that out of his three steps only one is backwards, the sample space for the experiment is:

$$\{(F,F,B);(F,B,F);(B,F,F)\}$$

where F denotes a step forwards, and B a step backwards. Each of these sample points is equally likely, therefore the probability that his first step is a step forward is $\frac{2}{3}$.

2. Let B_j denote the event that the ball is in box j and S_i denote the event that a search in box i uncovers the ball. We are given that $\mathbf{P}(B_j) = P_j$, $\mathbf{P}(S_i|B_i) = \alpha_i$ and we wish to compute the probability $\mathbf{P}(B_j|S_i^c)$ for each $i, j = 1, 2, \ldots, n$. Note also that $\mathbf{P}(S_i|B_i^c) = 0$. Using Bayes' Rule and the Total Probability Theorem,

$$\begin{aligned}
\mathbf{P}(B_j|S_i^c) &= \frac{\mathbf{P}(S_i^c|B_j)\mathbf{P}(B_j)}{\mathbf{P}(S_i^c)} = \frac{\mathbf{P}(S_i^c|B_j)\mathbf{P}(B_j)}{\mathbf{P}(S_i^c|B_i)\mathbf{P}(B_i) + \mathbf{P}(S_i^c|B_i^c)\mathbf{P}(B_i^c)} \\
&= \frac{\mathbf{P}(S_i^c|B_j)P_j}{(1-\alpha_i)P_i + 1(1-P_i)} = \frac{\mathbf{P}(S_i^c|B_j)P_j}{1-\alpha_i P_i}
\end{aligned}$$

Finally, the desired result is obtained when we also recognize that

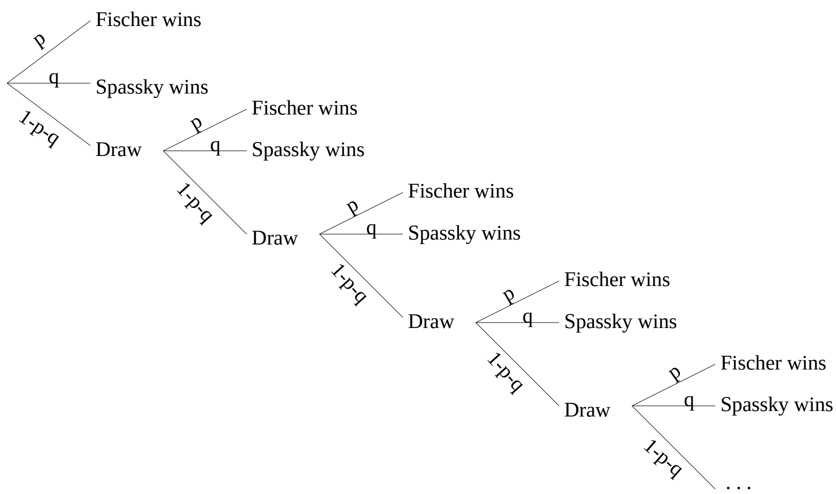
$$\mathbf{P}(S_i^c|B_j) = \begin{cases} 1-\alpha_i & , \quad j = i \\ 1 & , \quad j \neq i \end{cases} .$$

3. (a)

$$\begin{aligned}
\mathbf{P}(\text{Fischer wins}) &= p + p(1-p-q) + p(1-p-q)^2 + \cdots \\
&= \frac{p}{1-(1-p-q)} \\
&= \boxed{\frac{p}{p+q}}
\end{aligned}$$

We may also find the solution through a simpler method:

$$\begin{aligned}
\mathbf{P}(\text{Fischer wins} \mid \text{Someone wins}) &= \frac{\mathbf{P}(\text{Fischer wins})}{\mathbf{P}(\text{Someone wins})} \\
&= \boxed{\frac{p}{p+q}}
\end{aligned}$$



(b) $\mathbf{P}$(the match lasted no more than 5 games)
$= (p+q)+(p+q)(1-p-q)+(p+q)(1-p-q)^2+(p+q)(1-p-q)^3+(p+q)(1-p-q)^4$
$= \frac{(p+q)[1-(1-p-q)^5]}{1-(1-p-q)}$
$= 1-(1-p-q)^5$

$\mathbf{P}$(Fischer wins in the first game $\cap$ the match lasted no more than 5 games)
$= p$

Therefore, $\mathbf{P}$(Fischer wins | the match lasted no more than 5 games)
$= \frac{\mathbf{P}(\text{Fischer wins} \cap \text{the match lasted no more than 5 games})}{\mathbf{P}(\text{the match lasted no more than 5 games})}$
$= \boxed{\frac{p}{1-(1-p-q)^5}}$

(c) $\mathbf{P}$(the match lasted no more than 5 games)
$= 1-(1-p-q)^5$

$\mathbf{P}$(Fischer wins $\cap$ the match lasted no more than 5 games)
$= p+p(1-p-q)+p(1-p-q)^2+p(1-p-q)^3+p(1-p-q)^4$
$= \frac{p[1-(1-p-q)^5]}{1-(1-p-q)}$
$= \frac{p[1-(1-p-q)^5]}{p+q}$

Therefore, $\mathbf{P}$(Fischer wins | the match lasted no more than 5 games)
$= \frac{\mathbf{P}(\text{Fischer wins} \cap \text{the match lasted no more than 5 games})}{\mathbf{P}(\text{the match lasted no more than 5 games})}$
$= \boxed{\frac{p}{p+q}}$

(d) $\mathbf{P}$(Fischer wins at or before the 5th game | Fischer wins)
$= \frac{\mathbf{P}(\text{Fischer wins at or before the 5th game} \cap \text{Fischer wins})}{\mathbf{P}(\text{Fischer wins})}$
$= \left(\frac{p[1-(1-p-q)^5]}{p+q}\right) / \left(\frac{p}{p+q}\right)$
$= \boxed{1-(1-p-q)^5}$

This part may be solved by observing that the events {Fischer wins} and {the match lasted no more than 5 games} are independent (we know this from parts (a) and (c)):
$\mathbf{P}$(the match lasted no more than 5 games | Fischer wins)
$=$ $\mathbf{P}$(the match lasted no more than 5 games)
$= \boxed{1-(1-p-q)^5}$

4. We know that:

$$\mathbf{P}(A) = \mathbf{P}(A \cap B) + \mathbf{P}(A \cap B^c)$$

and therefore

$$\begin{aligned} \mathbf{P}(A \cap B^c) &= \mathbf{P}(A) - \mathbf{P}(A \cap B) \\ &= \mathbf{P}(A) - \mathbf{P}(A) \cdot \mathbf{P}(B) \\ &= \mathbf{P}(A)(1 - \mathbf{P}(B) \\ &= \mathbf{P}(A)\mathbf{P}(B^c) \end{aligned}$$

This proves (a) and (b).
For c, we use DeMorgan's Law:

$$\begin{aligned} \mathbf{P}(A \cap B) &= \mathbf{P}((A^c \cup B^c)^c) \\ &= 1 - \mathbf{P}(A^c \cup B^c) \\ &= 1 - \mathbf{P}(A^c) - \mathbf{P}(B^c) + \mathbf{P}(A^c \cap B^C) \end{aligned}$$

but we also have:

$$\begin{aligned} \mathbf{P}(A \cap B) &= \mathbf{P}(A) \cdot \mathbf{P}(B) \\ &= (1 - \mathbf{P}(A^c)) \cdot (1 - \mathbf{P}(B^c)) \\ &= 1 - \mathbf{P}(B^c) - \mathbf{P}(A^c) + \mathbf{P}(A^c) \cdot \mathbf{P}(B^c) \end{aligned}$$

This concludes the proof.

5. Without prior bias on whether the exit of campus lies East or West, the exact answers of the passerby are not as important as whether a string of answers is similar or not. Let R_r denote the event that we receive r similar answers and T denote the event that these repeated answers are truthful. Let S denote the event that the questioned passerby is a student. Note that, because a professor always gives a false answer, $T \cap S^c = \emptyset$ and thus $\mathbf{P}(T \cap S^c) = 0$. Therefore,

$$\mathbf{P}(T|R_r) = \frac{\mathbf{P}(T \cap R_r)}{\mathbf{P}(R_r)} = \frac{\mathbf{P}(T \cap R_r \cap S)}{\mathbf{P}(R_r)} = \frac{\mathbf{P}(T \cap R_r|S)\mathbf{P}(S)}{\mathbf{P}(R_r)}$$

where the stated independence of a passerby's successive answers implies $\mathbf{P}(T \cap R_r|S) = \left(\frac{3}{4}\right)^r$. Applying the Total Probability Theorem and again making use of independence, we also deduce

$$\mathbf{P}(R_r) = \mathbf{P}(R_r|S)\mathbf{P}(S) + \underbrace{\mathbf{P}(R_r|S^c)}_{1}\mathbf{P}(S^c) = \left(\left(\frac{3}{4}\right)^r + \left(\frac{1}{4}\right)^r\right)\frac{2}{3} + \frac{1}{3} \quad .$$

(a) Applying the above formulas for $r = 1$, we have $\mathbf{P}(R_1) = 1$ and thus

$$\mathbf{P}(T|R_1) = \frac{\frac{3}{4} \cdot \frac{2}{3}}{1} = \boxed{\frac{1}{2}} \quad .$$

(b) For $r = 2$, the formulas yield $\mathbf{P}(R_2) = \frac{3}{4}$ and thus

$$\mathbf{P}(T|R_2) = \frac{\left(\frac{3}{4}\right)^2 \frac{2}{3}}{\frac{3}{4}} = \boxed{\frac{1}{2}}.$$

(c) For $r = 3$, the formulas yield $\mathbf{P}(R_3) = \frac{15}{24}$ and thus

$$\mathbf{P}(T|R_3) = \frac{\left(\frac{3}{4}\right)^3 \frac{2}{3}}{\frac{15}{24}} = \boxed{\frac{9}{20}} \quad .$$

(d) For $r = 4$, the formulas yield $\mathbf{P}(R_4) = \frac{35}{64}$ and thus

$$\mathbf{P}(T|R_4) = \frac{\left(\frac{3}{4}\right)^4 \frac{2}{3}}{\frac{35}{64}} = \boxed{\frac{27}{70}} \quad .$$

(e) As soon as we receive a dissimilar answer from the same passerby, we know that this passerby is a student; a professor will always give the same (false) answer. Let D denote the event of receiving the first dissimilar answer. Given D on the fourth answer, either the student has provided three truthful answers followed by one untruthful answer, occurring with probability $\left(\frac{3}{4}\right)^3 \frac{1}{4}$, or the student has provided three untruthful answers followed by one truthful answer, occurring with probability $\left(\frac{1}{4}\right)^3 \frac{3}{4}$. Note that event T corresponds to the former; thus,

$$\mathbf{P}(T|R_3 \cap D) = \frac{\left(\frac{3}{4}\right)^3 \frac{1}{4}}{\left(\frac{3}{4}\right)^3 \frac{1}{4} + \left(\frac{1}{4}\right)^3 \frac{3}{4}} = \boxed{\frac{9}{10}} \quad .$$

In parts (a) - (d), notice the decreasing trend in the probability of the passer-by being truthful as the number of similar answers grows. Intuitively, our confidence that the passerby is a professor grows as the sequence of similar answers gets longer, because we know a professor will always give the same (false) answer while a student has a chance to answer either way. However, as part (e) demonstrates, the first indication that the passerby is a student will boost our confidence that the previous string of similar answers are truthful, because any single answer by the student has a 3-to-1 chance of being a truthful one.

For the remainder of this problem, let E and W represent the events that a passerby provides East and West, respectively, as an answer and let T_E represent the event that East is the correct answer. We are told Ima's a-priori bias is $\mathbf{P}(T_E) = \epsilon$.

(f) Using Bayes's Rule and all the arguments used in parts (a) - (e), we have

$$\mathbf{P}(T_E|E) = \frac{\mathbf{P}(E|T_E)\mathbf{P}(T_E)}{\mathbf{P}(E)} = \frac{\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)\epsilon}{\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)\epsilon + \left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)+\frac{1}{3}\right)(1-\epsilon)} = \boxed{\epsilon} \quad \text{and}$$

$$\mathbf{P}(T_E|W) = \frac{\mathbf{P}(W|T_E)\mathbf{P}(T_E)}{\mathbf{P}(W)} = \frac{\left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)+\frac{1}{3}\right)\epsilon}{\left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)+\frac{1}{3}\right)\epsilon + \left(\frac{2}{3}\right)\left(\frac{3}{4}\right)(1-\epsilon)} = \boxed{\epsilon} \quad .$$

In particular, we have used that $\mathbf{P}(E) = \mathbf{P}(E|T_E)\mathbf{P}(T_E) + \mathbf{P}(E|T_E^c)\mathbf{P}(T_E^c)$ (and similarly for $\mathbf{P}(W)$)

(g) Likewise, given two consecutive and similar answers from the same passerby, we have

$$\mathbf{P}(T_E|EE) = \frac{\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)^2\epsilon}{\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)^2\epsilon + \left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)^2+\frac{1}{3}\right)(1-\epsilon)} = \boxed{\epsilon} \quad \text{and}$$

$$\mathbf{P}(T_E|WW) = \frac{\left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)^2+\frac{1}{3}\right)\epsilon}{\left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)^2+\frac{1}{3}\right)\epsilon + \left(\frac{2}{3}\right)\left(\frac{3}{4}\right)^2(1-\epsilon)} = \boxed{\epsilon} \quad .$$

(h) Finally, given three consecutive and similar answers from the same passerby,

$$\mathbf{P}(T_E|EEE) = \frac{\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)^3\epsilon}{\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)^3\epsilon + \left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)^3+\frac{1}{3}\right)(1-\epsilon)} = \boxed{\frac{9\epsilon}{11-2\epsilon}} \quad \text{and}$$

$$\mathbf{P}(T_E|WWW) = \frac{\left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)^3+\frac{1}{3}\right)\epsilon}{\left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)^3+\frac{1}{3}\right)\epsilon + \left(\frac{2}{3}\right)\left(\frac{3}{4}\right)^3(1-\epsilon)} = \boxed{\frac{11\epsilon}{9+2\epsilon}} \quad .$$

For $\epsilon = \frac{9}{20}$, we calculate $\mathbf{P}(T_E|EEE) = \frac{81}{202}$ and $\mathbf{P}(T_E|WWW) = \frac{1}{2}$.

Notice that the E, EE and EEE answers to parts (f) - (h) match the answers to parts (a)-(c) when $\epsilon = \frac{1}{2}$, or when Ima's prior bias does not favor either possibility.