Problem Set 4
Due: March 2, 2005

1. Consider an experiment in which a fair four-sided die (with faces labeled 0, 1, 2, 3) is thrown once to determine how many times a fair coin is to be flipped. In the sample space of this experiment, random variables N and K are defined by

   - N = the result of the die roll
   - K = the total number of heads resulting from the coin flips

   Determine and sketch each of the following functions for all values of their arguments:

   (a) $p_N(n)$
   (b) $p_{K|N}(k \mid 2)$
   (c) $p_{N|K}(n \mid 2)$
   (d) $p_K(k)$
   (e) Also determine the conditional PMF for random variable N, given that the experimental value of K is an odd number.
   (f) Finally, compute the expected value and variance for each of the distributions found in parts (a)–(e).

2. Random variables X and Y have the joint PMF

$$p_{X,Y}(x,y) = \begin{cases} c(x^2+y^2) & , \quad x \in \{1,2,4\} \quad \text{and} \quad y \in \{1,3\} \\ 0 & , \quad \text{otherwise.} \end{cases}$$

   (a) What is the value of the constant c?
   (b) What is $\mathbf{P}(Y < X)$?
   (c) What is $\mathbf{P}(Y > X)$?
   (d) What is $\mathbf{P}(Y = X)$?
   (e) What is $\mathbf{P}(Y = 3)$?
   (f) Find the marginal PMFs $p_X(x)$ and $p_Y(y)$.
   (g) Find the expectations $\mathbf{E}[X]$ and $\mathbf{E}[Y]$.
   (h) Find the variances $\text{var}(X)$ and $\text{var}(Y)$.

3. You just rented a large house and the realtor gave you five keys, one for the front door and the other four for each of the four side and back doors of the house. Unfortunately, all keys look identical, so to open the front door, you are forced to try them at random.

   Find the mean and the variance of the number of trials you will need to open the door, under the following alternative assumptions:

   (a) after an unsuccessful trial, you mark the corresponding key so that you never try it again, or
   (b) at each trial, you are equally likely to choose any key.

4. A contestant on a quiz show is presented with two questions, question 1 and 2, which he is to attempt to answer in some order chosen by him. If he decides to try question i, $i = 1, 2$ first, then he will be allowed to go on to question j, $j \neq i$, only if his answer to i is correct. If his initial answer is incorrect, he is not allowed to answer the other question. The contestant is to receive v_i dollars if he answers question i correctly, $i = 1, 2$. Thus, for instance, he will receive $v_1 + v_2$ dollars if both questions are correctly answered. If the probability that he knows the answer to question i is p_i, $i = 1, 2$, which question should he attempt first so as to maximize his expected winnings? Assume that the events E_i, $i = 1, 2$, that he knows the answer to question i, are independent events.