MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2005)

Recitation 22 Answers
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1. (Convergence Revisited)

   (a) $\mathbf{E}[X_n] = 0 * (1 - \frac{1}{n}) + 1 * \frac{1}{n} = \frac{1}{n}$
   $\text{var}(X_n) = (0 - 1/n)^2 * (1 - 1/n) + (1 - 1/n)^2 * (1/n) = \frac{n-1}{n^2}$
   $\mathbf{E}[Y_n] = 0 * (1 - \frac{1}{n}) + n * \frac{1}{n} = 1$
   $\text{var}(Y_n) = (0 - 1)^2 * (1 - 1/n) + (n - 1)^2 * (1/n) = n - 1$

   (b) Using the Chebyshev inequality, we have
   $\lim_{n->\infty} \mathbf{P}(|X_n - \frac{1}{n}| \geq \epsilon) \leq \lim_{n->\infty} \frac{n-1}{n^2\epsilon^2} = 0$
   Moreover, $\lim_{n->\infty} \frac{1}{n} = 0$.
   it follows that X_n is convergent in probability.

   (c) In this case, Chebyshev suggests that
   $\lim_{n->\infty} \mathbf{P}(|Y_n - 1| \geq \epsilon) \leq \lim_{n->\infty} \frac{n-1}{\epsilon^2} = \infty$,
   It follows that the Chebyshev inequality tells us nothing.

   (d) For every $\epsilon > 0$, we have
   $\lim_{n->\infty} \mathbf{P}(|X_n| \geq \epsilon) \leq \lim_{n->\infty} \frac{1}{n} = 0$,
   which suggests that X_n converges to zero in probability.

   (e) For every $\epsilon > 0$, we have
   $\lim_{n->\infty} \mathbf{P}(|Y_n| \geq \epsilon) \leq \lim_{n->\infty} \frac{1}{n} = 0$,
   which suggests that Y_n converges to zero in probability.

2. (a) In this part of the problem, we need a cumulative distribution function (CDF) for the sum of 102 independent experimental values of W, the weight of a pretzel. According to a central limit theorem discussed in class, we can approximate this CDF with the CDF for a Gaussian random variable with the same expectation and variance. If we define random variable R to be the sum of 102 independent experimental values of W, we have

   $$\mathbf{E}[R] = 102\mathbf{E}[W] \qquad \sigma_R^2 = 102\sigma_W^2 \quad \text{where} \quad \mathbf{P}(R \leq r) \approx \Phi\left(\frac{r - \mathbf{E}[R]}{\sigma_R}\right)$$

   We can find $\mathbf{E}[W]$ and σ_W^2 using the given PDF. By inspection, $\mathbf{E}[W] = 2$. We first find $\mathbf{E}[W^2]$ to calculate $\sigma_W^2 = \mathbf{E}[W^2] - \mathbf{E}[W]^2$:

   $$\mathbf{E}[W^2] \;=\; \int_0^\infty w^2 f_W(w)dw = \int_1^2 w^2(w-1)dw + \int_2^3 w^2(3-w)dw = \frac{25}{6} \Rightarrow \sigma_W^2 = \frac{1}{6}$$

   So $\mathbf{E}[R] = 102 \cdot 2 = 204$ and $\sigma_R^2 = 102 \cdot \frac{1}{6} = 17$. Using the CLT approximation we have

   $$\mathbf{P}(R > 200) \approx 1 - \Phi\left(\frac{200 - 204}{\sqrt{17}}\right) = 1 - \left[1 - \Phi\left(\frac{204 - 200}{\sqrt{17}}\right)\right] = \Phi(0.9701) \approx \boxed{.8340}$$

   (b) We are trying to find the smallest value of N such that $\mathbf{P}(R > 200) = .990$. We will again use a CLT approximation, but this time $\mathbf{E}[R] = N\mathbf{E}[W] = 2N$ and $\sigma_R^2 = N\sigma_W^2 = \frac{N}{6}$. Since we want the weight of N pretzels to exceed 200 ounces with probability .990, we have

   $$1 - \mathbf{P}(R \leq 200) = 0.990 \quad \Rightarrow \quad 1 - \Phi\left(\frac{200 - 2N}{\sqrt{\frac{N}{6}}}\right) \approx .990 \quad \Rightarrow \quad \Phi\left(\frac{200 - 2N}{\sqrt{\frac{N}{6}}}\right) \approx .01$$

When $\Phi(y_0) < 0.5$, then y_0 is negative. Furthermore, because $\Phi(-|y_0|) = 1 - \Phi(|y_0|)$, we have

$$1 - \Phi\left(\frac{-200+2N}{\sqrt{\frac{N}{6}}}\right) \approx .01 \quad \Rightarrow \quad \Phi\left(\frac{-200+2N}{\sqrt{\frac{N}{6}}}\right) \approx .990 \quad \Rightarrow \quad \frac{-200+2N}{\sqrt{\frac{N}{6}}} \approx 2.33$$

So,

$$\begin{aligned} 40000 - 800N + 4N^2 \approx \frac{2.33^2}{6}N \quad &\Rightarrow \quad 4N^2 - 800.905N + 40000 \approx 0 \\ &\Rightarrow \quad N \approx 104.87 \text{ or } 95.36 \end{aligned}$$

Which value is correct? Consider $\mathbf{E}[R]$ for each possible value of N. $\mathbf{E}[R] = 2N$ must be greater than 200 for the question to make sense, as we showed above that $y_0 = (200 - 2N)/\sigma_R < 0$. The value that achieves this is $N = 104.87$. This number corresponds to the amount of pretzels we need so that their total weight is 200 ounces with probability .990. Therefore, to find the *smallest integer* for which the total weight *exceeds* 200 ounces with probability .990, we round *up* to $\boxed{105.}$

3. (a) When using just one mold, the length of the path is $25X$ and the desired probability is

$$\mathbf{P}(|25X - 1000| < 7.5) \;=\; \mathbf{P}(|X - 40| < 0.3) \;=\; \frac{\sqrt{3}}{10} \;\approx\; 0.1732.$$

(b) When using separate molds with lengths $X_1, X_2, \ldots, X_{25}$ the desired probability is

$$\begin{aligned} \mathbf{P}\left(\left|\left(\sum_{i=1}^{25} X_i\right) - 1000\right| < 7.5\right) \;&=\; \mathbf{P}\left(\frac{|(\sum_{i=1}^{25} X_i) - 1000|}{\sqrt{25}} < \frac{7.5}{\sqrt{25}}\right) \\ &\approx\; \mathbf{P}|Z| < 1.5 \qquad \text{where } Z \text{ is a standard normal r.v. (CLT)} \\ &=\; \Phi(1.5) - \Phi(-1.5) \;=\; 2\Phi(1.5) - 1 \approx 0.8664. \end{aligned}$$

Intuitively, adding independent instances of mold lengths “averages out” the variations and gives higher probability of a total path length close to the mean.