Recitation 14 Solutions
April 5, 2005

1. (a) $\mathbf{P}(X_1 = 1) = pq$. Therefore, X is a binomial random variable with parameter pq.

$p_{X_1}(0) = 1 - pq$
$p_{X_1}(1) = pq$

$p_X(0) = (1-pq)^3$
$p_X(1) = 3pq(1-pq)^2$
$p_X(2) = 3(pq)^2(1-pq$
$p_X(3) = (pq)^3$

(b)

$p_{Y_1}(0) = \mathbf{P}(\text{no bus arrives at time 1}) + \mathbf{P}(\text{a bus arrives and it heads to Kendall}) = (1-p)+pq$

$$p_{X_1|Y_1}(0\,|\,0) = \frac{p_{X_1,Y_1}(0\,,\,0)}{p_{Y_1}(0)} = \frac{1-p}{(1-p)+pq}$$

$$p_{X_1|Y_1}(1\,|\,0) = \frac{p_{X_1,Y_1}(1\,,\,0)}{p_{Y_1}(0)} = \frac{pq}{(1-p)+pq}$$

X_1 and Y_2 are independent. So

$$p_{X_1|Y_2}(0\,|\,0) = p_{X_1|Y_2}(0\,|\,1) = p_{X_1}(0) = 1 - pq$$

$$p_{X_1|Y_2}(1\,|\,0) = p_{X_1|Y_2}(1\,|\,1) = p_{X_1}(1) = pq$$

Therefore

$p_{X_1|Y_1}(0\,|\,0) = \frac{1-p}{(1-p)+pq}$

$p_{X_1|Y_1}(1\,|\,0) = \frac{pq}{(1-p)+pq}$

$p_{X_1|Y_1}(0\,|\,1) = 1$

$p_{X_1|Y_1}(1\,|\,1) = 0$

$p_{X_1|Y_2}(0\,|\,0) = 1 - pq$

$p_{X_1|Y_2}(1\,|\,0) = pq$

$p_{X_1|Y_2}(0\,|\,1) = 1 - pq$

$p_{X_1|Y_2}(1\,|\,1) = pq$

(c) X_i is not independent of Y_i for $i = 1, 2, 3$. If $Y_1 = 1$ then $X_1 = 0$ with probability 1. X_i is independent of Y_j, for $i, j = 1, 2, 3$ and $i \neq j$ because bus arrivals at any particular hour are independent of what happens at any other hour.

(d) Note that X_1 and Y_2 are independent. X_1 and Y_3 are independent.

- $p_{X_1|A}(0) = p_{X_1|Y_1}(0\,|\,0) = \frac{p_{X_1,Y_1}(0\,,0)}{p_{Y_1}(0)} = \frac{1-p}{(1-p)+pq}$

- $p_{X_1|A}(1) = p_{X_1|Y_1}(1\,|\,0) = \frac{p_{X_1,Y_1}(1\,,0)}{p_{Y_1}(0)} = \frac{pq}{(1-p)+pq}$

(e)

$$\{Y=2\} = \begin{cases} A: \{Y_1=1, Y_2=1, Y_3=0\}, \\ B: \{Y_1=1, Y_2=0, Y_3=1\}, \\ C: \{Y_1=0, Y_2=1, Y_3=1\}, \end{cases}$$

Using the total probability theorem,

$$p_{X_1|Y}(0\,|\,2) = p_{X_1|Y_1}(0\,|\,1)[\mathbf{P}(A)+\mathbf{P}(B)] + p_{X_1|Y_1}(0\,|\,0)\mathbf{P}(C) = 1\cdot\frac{2}{3} + \frac{1-p}{(1-p)+pq}\cdot\frac{1}{3}$$

and

$$p_{X_1|Y}(1\,|\,2) = p_{X_1|Y_1}(1\,|\,1)[\mathbf{P}(A)+\mathbf{P}(B)] + p_{X_1|Y_1}(1\,|\,0)\mathbf{P}(C) = 0\cdot\frac{2}{3} + \frac{pq}{(1-p)+pq}\cdot\frac{1}{3}.$$

Therefore

- $p_{X_1|Y}(0\,|\,2) = \frac{2}{3} + \frac{1}{3}\cdot\frac{1-p}{(1-p)+pq}$

- $p_{X_1|Y}(1\,|\,2) = \frac{1}{3}\cdot\frac{pq}{(1-p)+pq}$

2. See online solutions.

3. Define the following events:

Event R: Robber attempts robbery.
Event S: Robbery is successful.

Define the following random variables:

X: The number of days up to and including the first successful robbery.
B: The number of candy bars stolen during a successful robbery.
C: The number of days of rest after a successful robbery.

Note that X is a geometric random variable with parameter $\frac{3}{20}$ (the probability of a successful robbery on a given night). Also, it is given that B is uniform over $\{1,2,3\}$ with probability $\frac{1}{3}$ each, and that C is uniform over $\{2,4\}$ with probability $\frac{1}{2}$ each.

(a) We will derive first the PMF and then the transform of D, the number of days up to and including the second successful robbery.

The PMF of D can be easily found by conditioning on C, the number of days the robber rests after the first successful robbery (which only takes on values 2 or 4):

$$p_D(d) = p_{D|C}(d\mid 2)\mathbf{P}(C=2) + p_{D|C}(d\mid 4)\mathbf{P}(C=4),$$

where

$$p_{D|C}(d\mid 2) = \begin{cases} \binom{d-3}{1}(\frac{3}{20})(\frac{17}{20})^{d-4}(\frac{3}{20}) & \text{if } d=4,5,6,\ldots \\ 0 & \text{otherwise.} \end{cases}$$

There must be at least one day before the rest period, since it follows a successful robbery; similarly, there must be at least one day after the rest period. Thus we can view the coefficient in the preceding formula as the number of ways to choose the beginning of a four-day period in a block of d days. Then we multiply the probability of the first success and the probability of $d-4$ failures and finally the probability of the second success at trial d.

Similarly, we have:

$$p_{D|C}(d \mid 4) = \begin{cases} \binom{d-5}{1}(\frac{3}{20})(\frac{17}{20})^{d-6}(\frac{3}{20}) & \text{if } d = 6,7,8... \\ 0 & \text{otherwise.} \end{cases}$$

Also, $P(C=2) = P(C=4) = \frac{1}{2}$. So plugging into our expression for $p_D(d)$, we get

$$p_D(d) = \begin{cases} \frac{1}{2}\binom{d-3}{1}(\frac{3}{20})^2(\frac{17}{20})^{d-4} & \text{if } d = 4,5 \\ \frac{1}{2}\binom{d-3}{1}(\frac{3}{20})^2(\frac{17}{20})^{d-4} + \frac{1}{2}\binom{d-5}{1}(\frac{3}{20})^2(\frac{17}{20})^{d-6} & \text{if } d \geq 6 \\ 0 & \text{otherwise.} \end{cases}$$

Alternately, we can solve this problem using transforms. Note that D is the sum of three independent random variables: X_1, the number of days until the first successful robbery; C, the number of days of rest after the first success; and X_2, the number of days from the end of the rest period until the second successful robbery. Since $D = X_1 + C + X_2$, and X_1, X_2, and C are mutually independent, we find that the transform of D is $M_D(s) = [M_X(s)]^2 M_C(s)$.

Now, X_1 and X_2 are (independent) geometric random variables with parameter $\frac{3}{20}$ (the probability of a successful robbery). Again, C is equally likely to be 2 or 4. Thus we conclude that

$$M_D(s) = [M_X(s)]^2 M_C(s) = \left[\frac{\frac{3}{20}e^s}{1-\frac{17}{20}e^s}\right]^2 [\tfrac{1}{2}e^{2s} + \tfrac{1}{2}e^{4s}].$$

(b) Observe that since the probability of a robbery attempt being successful is $\frac{3}{4}$, with the number of candy bars taken in a successful attempt equally likely to be 1, 2, or 3, we can view each attempt as resulting in B candy bars, with the following PMF for B:

$$p_B(b) = \begin{cases} \frac{1}{4} & \text{if } b = 0,1,2,3, \\ 0 & \text{otherwise.} \end{cases}$$

Now, let T, U, and V be the number of attempts that result in 1, 2, and 3 candy bars, respectively. Then if a total of Q candy bars are stolen during the ten robbery attempts, $0 \leq V \leq \frac{Q}{3}$, $0 \leq U \leq \frac{Q-3V}{2}$, and $0 \leq T \leq Q - 3V - 2U$, with exactly $10 - T - U - V$ attempts failing. Thus the PMF for Q is

$$p_Q(q) = \sum_{v=0}^{q/3} \sum_{u=0}^{\frac{q-3v}{2}} \sum_{t=0}^{q-3v-2u} \binom{10}{t, u, v, 10-t-u-v}, \quad \text{for } 0 \leq q \leq 30.$$

Of course, since Q is the sum of ten independent values of B, the transform is simply $M_Q(s) = [M_B(s)]^{10} = (1 + e^s + e^{2s} + e^{3s})^{10}/4^{10}$.

Observe that since $Q = B_1 + \cdots + B_{10}$, $\mathbf{E}[Q] = E[B_1] + \cdots + E[B_{10}] = 10 \cdot \mathbf{E}[B] = 15$. Similarly, using the independence of the $B_i's$, we see that $\text{Var}(Q) = \text{Var}(B_1) + \cdots + \text{Var}(B_{10}) = 10 \cdot \text{Var}(B) = 10 \cdot \frac{5}{4} = \frac{50}{4}$.