Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2005)

Final Exam Review Solutions
May 13, 2005

1. (a) We know that the total length of the edge for red interval is two times that for black inteval. Since the ball is equally likely to fall in any position of the edge, probability of falling in a red interval is $\frac{2}{3}$.

   (b) Conditioned on the ball having fallen in a black interval, the ball is equally likely to fall anywhere in the interval. Thus, the PDF is

$$f_{Z|\text{black interval}}(z) = \begin{cases} \frac{15}{\pi r} & , \quad z \in [0, \frac{\pi r}{15}] \\ 0 & , \quad \text{otherwise} \end{cases}$$

   (c) Since the ball is equally likely to fall on any point of the edge, we can see it is twice as likely for $z \in [0, \frac{\pi r}{15})$ than $z \in \left[\frac{\pi r}{15}, \frac{2\pi r}{15}\right]$. Therefore, intuitively, let

$$f_Z(z) = \begin{cases} 2h & , \quad z \in [0, \frac{\pi r}{15}) \\ h & , \quad z \in \left[\frac{\pi r}{15}, \frac{2\pi r}{15}\right] \\ 0 & , \quad \text{otherwise} \end{cases}$$

   Using the fact that $\int_{-\infty}^{\infty} f_Z(z) = 1$,

$$(2h)(\frac{\pi r}{15}) + (h)(\frac{\pi r}{15}) = 1 \Rightarrow h = \frac{5}{\pi r}$$

$$f_Z(z) = \begin{cases} \frac{10}{\pi r} & , \quad z \in [0, \frac{\pi r}{15}) \\ \frac{5}{\pi r} & , \quad z \in \left[\frac{\pi r}{15}, \frac{2\pi r}{15}\right] \\ 0 & , \quad \text{otherwise} \end{cases}$$

   (d) The total gains (or losses), T, equals to the sum of all X_i, i.e. $T = X_1 + X_2 + \cdots + X_n$, Since all the X_i's are independent of each other, and they have the same Gaussian distribution, the sum will also be a Gaussian with

$$\mathbf{E}[T] = \mathbf{E}[X_1] + \mathbf{E}[X_2] + \cdots + \mathbf{E}[X_n] = 0$$

$$\text{var}(T) = \text{var}(X_1) + \text{var}(X_2) + \cdots + \text{var}(X_n) = n\sigma^2$$

   Therefore, the standard deviation for T is $\sqrt{n}\sigma$.

   (e)

$$\begin{aligned} \mathbf{P}(|T| > 2\sqrt{n}\sigma) &= \mathbf{P}(T > 2\sqrt{n}\sigma) + \mathbf{P}(T < -2\sqrt{n}\sigma) \\ &= 2\mathbf{P}(T > 2\sqrt{n}\sigma) \\ &= 2\left(1 - \Phi\left(\frac{2\sqrt{n}\sigma - \mathbf{E}[T]}{\sigma_T}\right)\right) \\ &= 2(1 - \Phi(2)) \simeq 0.0454. \end{aligned}$$

2. **(a)** Let K = the number of arrivals in the three-hour period in consideration, and M_i = the size of the ith arrival group, which includes the user of the mailbox plus his or her accompanying friends. Thus, $M_i = N_i + 1$. We can express Y as

$$Y = M_1 + M_2 + \cdots + M_K.$$

Y is a random sum of random variables. Using iterated expectations:

$$\mathbf{E}[Y] = \mathbf{E}[\mathbf{E}[Y|K]] = \mathbf{E}[K]\mathbf{E}[M] = \mathbf{E}[K](\mathbf{E}[N] + 1).$$

Using the moment-generating property of transforms:

$$\begin{aligned}\mathbf{E}[N] &= \left.\frac{dM_N(s)}{ds}\right|_{s=0} \\ &= 1/3 + 2/6 \\ &= 2/3,\end{aligned}$$

K is Poisson with rate λ over an interval of length 3, so $\mathbf{E}[K] = 3\lambda$. Combining everything:

$$\boxed{\mathbf{E}[Y] = 5\lambda}$$

(b) We use the Pascal PMF:

$$\begin{aligned}&\mathbf{P}(\text{Exactly 3 parcels arrive by the time the 5th letter arrives}) \\ =\ &\mathbf{P}(\text{the fifth letter arrives at time 8}) \\ =\ &\mathbf{P}(\text{the first 7 arrivals consists of exactly 3 parcels, and 4 letters, } \textit{and} \text{ the 8th arrival is a letter}) \\ =\ &\binom{7}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^4\frac{2}{3} \\ =\ &\boxed{\binom{7}{3}\left(\tfrac{1}{3}\right)^3\left(\tfrac{2}{3}\right)^5}\end{aligned}$$

(c) We use the binomial PMF:

$$\begin{aligned}&\mathbf{P}(\text{Exactly 3 out of next 8 users will mail parcels}) \\ =\ &\boxed{\binom{8}{3}\left(\tfrac{1}{3}\right)^3\left(\tfrac{2}{3}\right)^5}\end{aligned}$$

(d) Let M be the size of the group to which any person belongs. $M = N + 1$, the number of accompanying friends plus the person himself).

$$M_M(s) = \mathbf{E}[e^{s(N+1)}] = e^s M_N(s) = \frac{1}{2}e^s + \frac{1}{3}e^{2s} + \frac{1}{6}e^{3s}$$

Note that, using part b), $\mathbf{E}[M] = \mathbf{E}[N] + 1 = 5/3$. Also $p_M(1) = \frac{d}{de^s}M_M(s)|_{e^s=0} = 1/2$. The key observation to this problem is that we are actually dealing with *random incidence*.

Imagine we line everyone up (both users and non-users), with people coming from the same group standing next to their accompanying friends. Suppose now that we pick a person i at random from this line. Let W be the size of the group to which i belongs. The probability that $W = w$ (i.e., the probability that she is from a group of size w) is the same as the probability that the gap into which we enter a process by random incidence is of duration w. The probability that a group of a certain size, w, is chosen is

$$p_W(w) = \frac{w p_M(w)}{\mathbf{E}[M]}$$

In particular, $p_W(1)$ is the probability that a randomly selected person i is the only person in the group, i.e., he or she is unaccompanied. Therefore:

$$\begin{aligned} & \mathbf{P}(\text{a randomly selected person is accompanied}) \\ = \;& 1 - \mathbf{P}(\text{a randomly selected person is } \textit{not} \text{ accompanied}) \\ = \;& 1 - p_W(1) \\ = \;& \boxed{7/10}. \end{aligned}$$

(e) As in part (a), define Y = the total number of people arriving in one hour, K = the number of arrivals in *one* hour, and Q_i = the size of the ith arrival group. We can express Y as:

$$Y = Q_1 + Q_2 + \cdots + Q_K.$$

where the Q_i's are independent and identically distributed. The transform of Y is:

$$\begin{aligned} M_Y(s) &= \mathbf{E}[e^{sY}] \\ &= \mathbf{E}[e^{sQ_1} e^{sQ_2} \dots e^{sQ_K}] \\ &= \mathbf{E}[\mathbf{E}[e^{sQ_1} e^{sQ_2} \dots e^{sQ_K} | K]] \\ &= \mathbf{E}[(M_Q(s))^K] \\ &= M_K(s)|_{e^s = M_Q(s)} \\ &= M_K(s)|_{e^s = \frac{1}{2}e^s + \frac{1}{3}e^{2s} + \frac{1}{6}e^{3s}} \end{aligned} \tag{1}$$

K is a Poisson random variable with parameter λ, so:

$$M_K(s) = e^{\lambda(e^s - 1)}.$$

Plugging in:

$$\boxed{M_Y(s) = e^{\lambda(\frac{1}{2}e^s + \frac{1}{3}e^{2s} + \frac{1}{6}e^{3s} - 1)}}$$

To get the actual closed-form PMF, if it exists at all, one must invert the transform $M_Y(s)$ above – a very difficult task. The lesson here is that for random sums of random variables it is usually much easier to get their transforms than their PMFs.

(f) Let us select a time in this Poisson process by random incidence. The time from our random entry to the arrival of the future fifth user is:

$$X_1 + \cdots + X_5$$

where the X_i's are i.i.d. exponential random variable with mean $1/\lambda$. Also, the time from the random entry to the previous third user would be:

$$X_{-3} + X_{-2} + X_{-1}$$

where each X_i's are once again independent exponential random variables with mean $1/\lambda$. So, T, the duration between the arrival of third previous users and the fifth future user would be

$$T = X_{-3} + X_{-2} + X_{-1} + X_1 + \cdots X_5$$

Hence, we have

$$\begin{aligned} M_T(s) &= (M_X(s))^8 \\ &= \boxed{\left(\frac{\lambda}{\lambda - s}\right)^8}. \end{aligned}$$

Note that $p_T(t)$ corresponds to the PDF of an Erlang random variable of order 8, i.e. the time until 8 arrivals in a Poisson process.

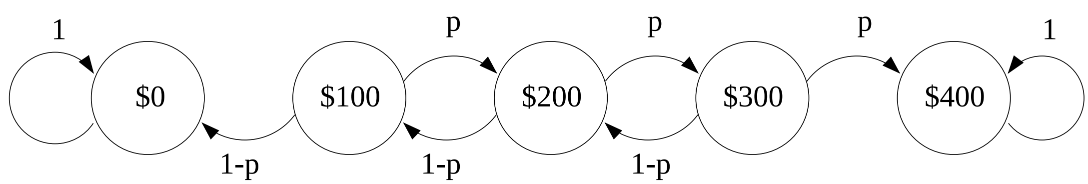
3. a) Assume Mary's goal is not to maximize her total profit but to simply maximize the probability she acquires \$400. We calculate the simple cases first. The easiest decision is when Mary has \$100. She must bet \$100 because she can't bet \$200.

   When Mary has \$300, she should bet \$100. Whether she bets \$100 or \$200, she will meet her goal if she wins the next game. If she bets \$100 and loses, she will then have \$200. If she bets \$200 and loses, she will then have \$100. Everything else being equal, it's more advantageous to have \$200 than to have \$100.

   The more difficult decision is how much to bet when Mary has \$200. We'll investigate both possible strategies and decide which is preferable.

   First, Mary can bet \$200. This case is easy to analyze. She will either win the desired amount or go "bust" on the next game. The probability that she will win is p.

   Second, Mary can bet \$100. In this case, we have the following state transition diagram.



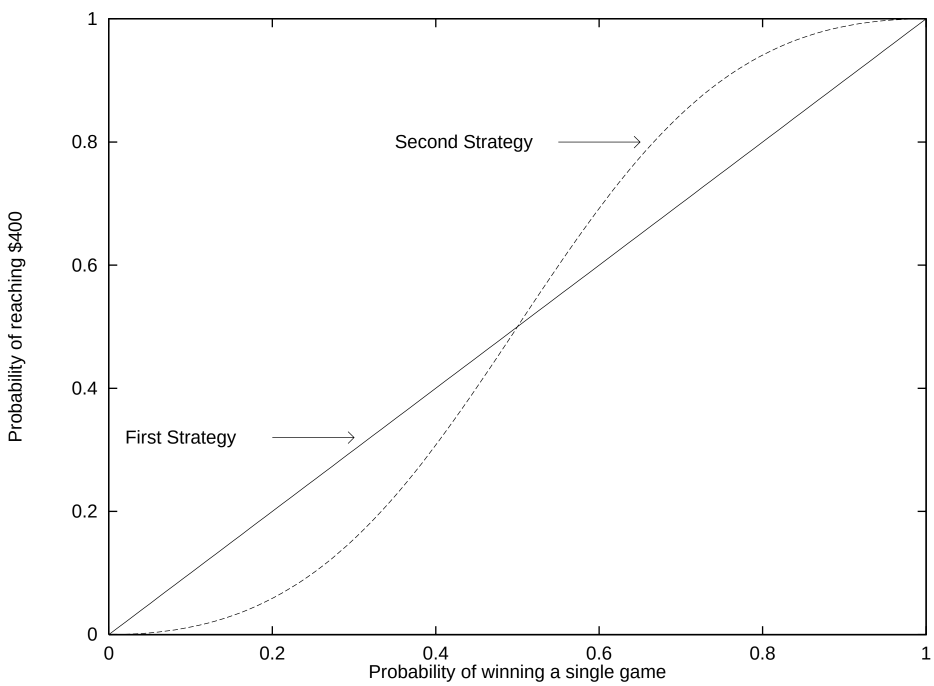
   Winning and going "bust" are the two absorption states, and we want to find the probability of eventually winning, given that she starts with \$200. We will denote the probability that Mary wins given that she starts with j hundred dollars by a_j.

$$\begin{aligned} a_1 &= a_2 p \\ a_2 &= a_1(1-p) + a_3 p \\ a_3 &= a_2(1-p) + p \end{aligned}$$

A few simple substitutions yield the following.

$$a_2 = \frac{p^2}{1 - 2p + 2p^2}$$

We need to compare p and a_2 to determine which strategy is optimal. Solving for the condition such that $p > a_2 = \frac{p^2}{1-2p+2p^2}$ yields that betting \$200 is advantageous when $p < 1/2$ and betting \$100 is advantageous when $p > 1/2$. When $p = 1/2$ neither strategy is better than the other.



b) With $p = .75$, the optimal strategy is for Mary to bet \$100 when she has \$200. We need to find the expected time until absorption, given that Mary started with \$200.

Let $\mu_i = \text{E}(\text{number of transitions to absorption starting from } i(\$100))$. We know that $\mu_0 = 0$ and $\mu_4 = 0$ because these are absorption states. We have the following relationship to determine the other μ_i's.

$$\mu_i = 1 + \sum_{j=1}^{3} p_{ij}\mu_j$$

So, we get the following three equations.

$$\begin{aligned}
\mu_1 &= 1 + p\mu_2 \\
\mu_2 &= 1 + (1-p)\mu_1 + p\mu_3 \\
\mu_3 &= 1 + (1-p)\mu_2
\end{aligned}$$

We need to solve for μ_2. Inserting $p = .75$, we get the following values for μ_i.

$$\begin{aligned}
\mu_1 &= 3.4 \\
\mu_2 &= 3.2 \\
\mu_3 &= 1.8
\end{aligned}$$

Therefore, the answer is $\boxed{3.2}$.