Recitation 7G Answers
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1. We note that

$$\begin{aligned}
\left(\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx\right)^2 &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-x^2/2} dx \int_{-\infty}^{+\infty} e^{-y^2/2} dy \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)/2} dx\ dy \\
&= \frac{1}{2\pi} \int_{0}^{2\pi} \int_{0}^{+\infty} e^{-r^2/2} r\ dr\ d\theta \\
&= \int_{0}^{+\infty} e^{-r^2/2} r\ dr \\
&= \int_{0}^{+\infty} e^{-u} du \\
&= -e^{-u}\big|_0^{+\infty} \\
&= 1,
\end{aligned}$$

where for the third equality, we use a transformation into polar coordinates, and for the fifth equality, we use the change of variables $u = r^2/2$. Thus, we have

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 1,$$

because the integral is positive. Using the change of variables $u = (x - \mu)/\sigma$, it follows that

$$\int_{-\infty}^{+\infty} f_X(x) dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du = 1.$$

2. This probem asks for basic use of the probability density function.

a) The probability that you wait more than 15 minutes is:

$$\int_{15}^{\infty} \frac{1}{15} e^{-\frac{x}{15}} dx = -e^{-\frac{x}{15}}\big|_{15}^{\infty} = e^{-1}.$$

b) The probability that you wait between 15 and thirty minutes is:

$$\int_{15}^{30} \frac{1}{15} e^{-\frac{x}{15}} dx = -e^{-\frac{x}{15}}\big|_{15}^{30} = e^{-1} - e^{-2}.$$

3. Given the following events,

A_t: Chip still works at time t.
B: The chip is bad.
G: The chip is good.

We can specify that $\mathbf{P}(A_t|G) = e^{-\alpha t}$ and $\mathbf{P}(A_t|B) = e^{-1000\alpha t}$.

(a) By using the definition of conditional probability and the total probability theorem, we can solve:

$$\mathbf{P}(A_t) = \mathbf{P}(G)\mathbf{P}(A_t|G) + \mathbf{P}(B)\mathbf{P}(A_t|B) = pe^{-\alpha t} + (1-p)e^{-1000\alpha t}$$

(b) By using the definition of conditional probability, we get:

$$\mathbf{P}(B|A_t) = \frac{\mathbf{P}(B \cap A_t)}{\mathbf{P}(A_t)}.$$

Furthermore, $\mathbf{P}(B \cap A_t) = \mathbf{P}(B)\mathbf{P}(A_t|B) = (1-p)e^{-1000\alpha t}$. Therefore,

$$\mathbf{P}(B|A_t) = \frac{(1-p)e^{-1000\alpha t}}{pe^{-\alpha t} + (1-p)e^{-1000\alpha t}}.$$

4. (a) Let G represent the event that the weather is good. We are given $\mathbf{P}(G) = \frac{2}{3}$.
To find the PDF of X, we first find the PDF of W, since $X = s + W = 2 + W$. We know that given good weather, $W \sim N(0,1)$. We also know that given bad weather, $W \sim N(0,9)$. To find the unconditional PDF of W, we use the density version of the total probability theorem.

$$\begin{aligned} f_W(w) &= \mathbf{P}(G) \cdot f_{W|G}(w) + \mathbf{P}(G^c) \cdot f_{W|G^c}(w) \\ &= \frac{2}{3} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} + \frac{1}{3} \cdot \frac{1}{3\sqrt{2\pi}} e^{-\frac{w^2}{2(9)}} \end{aligned}$$

We now perform a change of variables using $X = 2 + W$ to find the PDF of X:

$$f_X(x) = f_W(x-2) = \frac{2}{3} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-2)^2}{2}} + \frac{1}{3} \cdot \frac{1}{3\sqrt{2\pi}} e^{-\frac{(x-2)^2}{18}}.$$

(b) In principle, one can use the PDF determined in part (a) to compute the desired probability as

$$\int_1^3 f_X(x)\, dx.$$

It is much easier, however, to translate the event $\{1 \le X \le 3\}$ to a statement about W and then to apply the total probability theorem.

$$\mathbf{P}(1 \le X \le 3) = \mathbf{P}(1 \le 2 + W \le 3) = \mathbf{P}(-1 \le W \le 1)$$

We now use the total probability theorem.

$$\mathbf{P}(-1 \le W \le 1) \quad = \quad \mathbf{P}(G)\underbrace{\mathbf{P}(-1 \le W \le 1 \mid G)}_{a} + \mathbf{P}(G^c)\underbrace{\mathbf{P}(-1 \le W \le 1 \mid G^c)}_{b}$$

Since conditional on either G or G^c the random variable W is Gaussian, the conditional probabilities a and b can be expressed using Φ. Conditional on G, we have $W \sim N(0,1)$ so

$$a = \Phi(1) - \Phi(-1) = 2\Phi(1) - 1.$$

Conditional on G^c, we have $W \sim N(0, 9)$ so

$$b = \Phi\left(\frac{1}{3}\right) - \Phi\left(-\frac{1}{3}\right) = 2\Phi\left(\frac{1}{3}\right) - 1.$$

The final answer is thus

$$\mathbf{P}(1 \leq X \leq 3) = \frac{2}{3}\left(2\Phi\left(1\right) - 1\right) + \frac{1}{3}\left(2\Phi\left(\frac{1}{3}\right) - 1\right).$$