6.041 Quiz 2 Solutions
April 11, 2005

Problem 1: (55 points)

(a) (13 points)

$$\begin{aligned}\mathbf{E}[X] &= \mathbf{E}\left[\sum_{k=1}^{S} Z_k + Z_{S+1}T\right] = \mathbf{E}\left[\sum_{k=1}^{S} Z_k\right] + \mathbf{E}[Z_{S+1}]\mathbf{E}[T] = \mathbf{E}[Z_1]\mathbf{E}[S] + \mathbf{E}[Z_1]\mathbf{E}[T] \\ &= \mathbf{E}[Z_1](\mathbf{E}[S] + \mathbf{E}[T]) = \mathbf{E}[Z_1]\mathbf{E}[S+T] = \mathbf{E}[Z_1]\mathbf{E}[Y] = \mu.\end{aligned}$$

(b) (14 points)

$$\begin{aligned}p_S(k) &= \mathbf{P}(k \le Y < k+1) = F_Y(k+1) - F_Y(k) = (1 - e^{-(k+1)}) - (1 - e^{-k}) \\ &= e^{-k} - e^{-(k+1)} = e^{-k}(1 - e^{-1}).\end{aligned}$$

(c) (14 points) We have

$$f_{T|S}(t|k) = f_{Y|S}(t+k|k) = \begin{cases} \frac{f_Y(t+k)}{p_S(k)} & \text{if } k \le t+k < k+1, \\ 0 & \text{otherwise.} \end{cases}$$

Now, for $k \le t+k < k+1$, or equivalently, $0 \le t < 1$,

$$\frac{f_Y(t+k)}{p_S(k)} = \frac{e^{-(t+k)}}{e^{-k}(1-e^{-1})} = \frac{e^{-t}}{1-e^{-1}}.$$

Thus,

$$f_{T|S}(t|k) = \begin{cases} \frac{e^{-t}}{1-e^{-1}} & \text{if } 0 \le t < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(d) (14 points) We have

$$M_S(s) = \sum_{k=0}^{\infty} e^{sk} e^{-k}(1-e^{-1}) = (1-e^{-1}) \sum_{k=0}^{\infty} e^{(s-1)k} = (1-e^{-1}) \cdot \frac{1}{1-e^{s-1}}$$

and

$$M_{S+1}(s) = \mathbf{E}[e^{s(S+1)}] = e^s M_S(s) = (1-e^{-1}) \cdot \frac{e^s}{1-e^{s-1}}.$$

In addition,

$$M_Z(s) = e^{(\sigma^2 s^2/2)+\mu s}.$$

Therefore,

$$M_U(s) = (1-e^{-1}) \cdot \frac{e^{(\sigma^2 s^2/2)+\mu s}}{1-e^{(\sigma^2 s^2/2)+\mu s-1}}$$

and

$$M_V(s) = (1-e^{-1}) \cdot \frac{1}{1-e^{(\sigma^2 s^2/2)+\mu s-1}}.$$

Problem 2: (42 points)

(a) (14 points) We have $Z_t = X_t + Y_t$, so Z_t is a Gaussian-distributed random variable with mean vt and variance $\sigma_X^2 t + \sigma_Y^2$. Hence,

$$f_{Z_t}(z) = \frac{1}{\sqrt{2\pi(\sigma_X^2 t + \sigma_Y^2)}} e^{-(z-vt)^2/2(\sigma_X^2 t + \sigma_Y^2)}.$$

The correlation coefficient between Z_t and X_t is

$$\rho(Z_t, X_t) = \frac{\mathrm{cov}(Z_t, X_t)}{\sqrt{\mathrm{var}(Z_t)\mathrm{var}(X_t)}}.$$

Now,

$$\begin{aligned}
\mathrm{cov}(Z_t, X_t) &= \mathbf{E}[(Z_t - \mathbf{E}[Z_t])(X_t - \mathbf{E}[X_t])] \\
&= \mathbf{E}[(X_t - \mathbf{E}[X_t])(X_t - \mathbf{E}[X_t]) + (Y_t - \mathbf{E}[Y_t])(X_t - \mathbf{E}[X_t])] \\
&= \mathrm{var}(X_t) + \mathrm{cov}(Y_t, X_t) = \mathrm{var}(X_t).
\end{aligned}$$

Hence,

$$\rho(Z_t, X_t) = \sqrt{\frac{\mathrm{var}(X_t)}{\mathrm{var}(Z_t)}} = \sqrt{\frac{\sigma_X^2 t}{\sigma_X^2 t + \sigma_Y^2}}.$$

(b) (14 points) This question asks for the linear least-squares estimate of X_τ given $Z_\tau = z$, which is given by

$$\mathbf{E}[X_\tau] + \rho\frac{\sigma_{X_\tau}}{\sigma_{Z_\tau}}(Z_\tau - \mathbf{E}[Z_\tau]) = v\tau + \frac{\sigma_X^2\tau}{\sigma_X^2\tau + \sigma_Y^2}(z - v\tau) = \frac{\sigma_X^2\tau}{\sigma_X^2\tau + \sigma_Y^2}z + \frac{\sigma_Y^2}{\sigma_X^2\tau + \sigma_Y^2}v\tau.$$

(c) (14 points) The least-squares estimate of X_τ given $Z_\tau = z$ is given by

$$\mathbf{E}[X_\tau \mid Z_\tau = z] = \int_{-\infty}^{\infty} x f_{X_\tau|Z_\tau}(x|z)dx.$$

Now, using Bayes' rule,

$$\begin{aligned}
f_{X_\tau|Z_\tau}(x|z) &= \frac{f_{Z_\tau|X_\tau}(z|x)f_{X_\tau}(x)}{f_{Z_\tau}(z)} = \frac{f_{Y_\tau}(z-x)f_{X_\tau}(x)}{f_{Z_\tau}(z)} \\
&= \frac{\frac{1}{\sqrt{2\pi}\sigma_Y}e^{-(z-x)^2/2\sigma_Y^2}\frac{1}{\sqrt{2\pi\sigma_X^2\tau}}e^{-(x-v\tau)^2/2\sigma_X^2\tau}}{\frac{1}{\sqrt{2\pi(\sigma_X^2\tau+\sigma_Y^2)}}e^{-(z-v\tau)^2/2(\sigma_X^2\tau+\sigma_Y^2)}} \\
&= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\sigma_X^2\tau + \sigma_Y^2}{\sigma_X^2\sigma_Y^2\tau}}e^{-\frac{1}{2}\left(\frac{z^2-2xz+x^2}{\sigma_Y^2}+\frac{x^2-2v\tau x+v^2\tau^2}{\sigma_X^2\tau}\right)+\frac{(z-v\tau)^2}{2(\sigma_X^2\tau+\sigma_Y^2)}}.
\end{aligned}$$

Simplifying the exponent, we obtain

$$
\begin{aligned}
&-\frac{1}{2}\left(\frac{z^2-2xz+x^2}{\sigma_Y^2}+\frac{x^2-2v\tau x+v^2\tau^2}{\sigma_X^2\tau}\right)+\frac{(z-v\tau)^2}{2(\sigma_X^2\tau+\sigma_Y^2)} \\
&= -\frac{1}{2}\cdot\frac{\sigma_X^2\tau+\sigma_Y^2}{\sigma_X^2\sigma_Y^2\tau}\left[x^2-2x\left(\frac{\sigma_X^2\tau}{\sigma_X^2\tau+\sigma_Y^2}z+\frac{\sigma_Y^2}{\sigma_X^2\tau+\sigma_Y^2}v\tau\right)+\frac{\sigma_X^2\tau}{\sigma_X^2\tau+\sigma_Y^2}z+\frac{\sigma_Y^2}{\sigma_X^2\tau+\sigma_Y^2}v\tau\right] \\
&\quad+\frac{(z-v\tau)^2}{2(\sigma_X^2\tau+\sigma_Y^2)} \\
&= -\frac{1}{2}\cdot\frac{\sigma_X^2\tau+\sigma_Y^2}{\sigma_X^2\sigma_Y^2\tau}\left[x-\left(\frac{\sigma_X^2\tau}{\sigma_X^2\tau+\sigma_Y^2}z+\frac{\sigma_Y^2}{\sigma_X^2\tau+\sigma_Y^2}v\tau\right)\right]^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathbf{E}[X_\tau \mid Z_\tau = z] &= \int_{-\infty}^{\infty} x\frac{1}{\sqrt{2\pi}}\sqrt{\frac{\sigma_X^2\tau+\sigma_Y^2}{\sigma_X^2\sigma_Y^2\tau}}\,e^{-\frac{1}{2}\cdot\frac{\sigma_X^2\tau+\sigma_Y^2}{\sigma_X^2\sigma_Y^2\tau}\left[x-\left(\frac{\sigma_X^2\tau}{\sigma_X^2\tau+\sigma_Y^2}z+\frac{\sigma_Y^2}{\sigma_X^2\tau+\sigma_Y^2}v\tau\right)\right]^2}dx \\
&= \frac{\sigma_X^2\tau}{\sigma_X^2\tau+\sigma_Y^2}z+\frac{\sigma_Y^2}{\sigma_X^2\tau+\sigma_Y^2}v\tau,
\end{aligned}
$$

which is the same as the answer to part (b). It is also valid to argue that, because the only term in the exponent involving both x and z is of the form xz, the mean of the distribution of X_τ given $Z_\tau = z$ must be a linear function of z, which implies that the least-squares estimate is linear and therefore equal to linear least-squares estimate.

An alternative, and more general, proof is described in Section 4.7 of the textbook.