Problem Set 5: Solutions
Due: March 16, 2005

1. The question asks us to find a real number x such that:

$$\mathbf{P}(Z > x) \leq 0.01$$

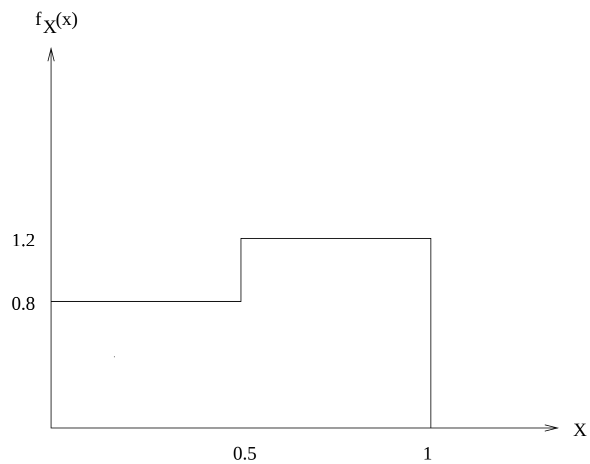
Now, for $0 \leq x \leq 1$

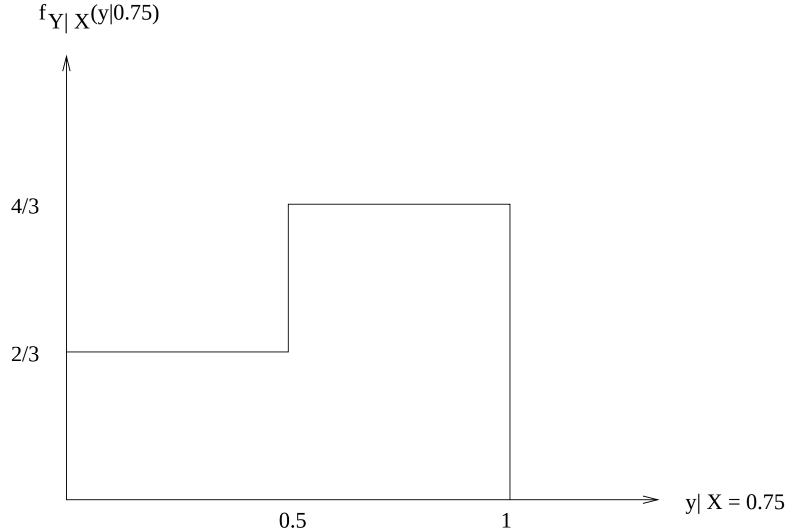
$$\begin{aligned}\mathbf{P}(x > x) &= 1 - \mathbf{P}(Z \leq x) \\ &= 1 - \int_0^x f_Z(z)dz \\ &= 1 - \int_0^x 5((1-z)^4) \\ &= 1 + (1-x)^5 - 1\end{aligned}$$

and from here we easily see that we need $x \geq 0.6019$.

2. (a) X and Y are not independent because there exist x and y such that $f_{X,Y}(x,y) \neq f_X(x)f_Y(y)$. For instance, $f_{X,Y}(\frac{2}{3},\frac{1}{3}) = 0.8$, $f_X(\frac{2}{3}) = \int_0^1 f_{X,Y}(\frac{2}{3},y)dy = 1.2$, $f_Y(\frac{1}{3}) = \int_0^1 f_{X,Y}(x,\frac{1}{3})dx = 0.8$, but $f_{X,Y}(\frac{2}{3},\frac{1}{3}) \neq f_X(\frac{1}{3})f_Y(\frac{1}{3})$.

(b) The plots are shown below.





$$f_X(x) = \begin{cases} 0.8, & 0 < x \leq 1/2 \\ 1.2, & 1/2 < x \leq 1 \\ 0, & \text{otherwise} \end{cases} \qquad f_{Y|X}(y|0.75) = \begin{cases} 2/3, & 0 < y \leq 1/2 \\ 4/3, & 1/2 < y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(c) Conditioned on event A, X and Y are independent. Thus

$$\mathbf{E}[R|A] = \mathbf{E}[XY|A] = \mathbf{E}[X|A]\mathbf{E}[Y|A] = (1/4)(1/2) = 1/8.$$

(d) It is easiest to see the CDF of W in this case as the integral of the PDF over an L-shaped area. For $0 < w \leq 1/2$ the CDF would be the integral over the PDF of the L-shaped area given by $((1)(w) + (w)(1-w))(0.8)$. Similarly, for $1/2 < w \leq 1$ the CDF would take on the values $(0.8)(3/4) + ((w-0.5)(0.5) + (1-w)(w-0.5))(1.6)$. Thus the entire CDF is given by

$$F_W(w) = \begin{cases} 0, & w \leq 0 \\ (2w - w^2)(0.8), & 0 < w \leq 1/2 \\ 1 - (1-w)^2(1.6), & 1/2 < w \leq 1 \\ 1, & w > 1 \end{cases}$$

3. We note that

$$\begin{aligned} \mathbf{P}(\text{error}) &= \mathbf{P}(\text{error} \,|\, \text{message sent is 1})\mathbf{P}(\text{message sent is 1}) + \\ &\quad \mathbf{P}(\text{error} \,|\, \text{message sent is 0})\mathbf{P}(\text{message sent is 0}). \end{aligned}$$

An error will occur when message 1 is sent if the received message is less than .5. This implies that $N < -1.5$. Therefore,

$$\begin{aligned} \mathbf{P}(\text{error} \,|\, \text{message sent is 1}) &= \mathbf{P}(N < -1.5) \\ &= 1 - \Phi(1.5) \\ &\approx .0668 \end{aligned}$$

Similarly, an error will occur when message 0 is sent if the received message is greater than or equal to .5. This implies that $N \geq 2.5$. Therefore,

$$\mathbf{P}(\text{error} \,|\, \text{message sent is 0}) = \mathbf{P}(N \geq 2.5)$$

$$\begin{aligned} &= 1 - \Phi(2.5) \\ &\approx .0062 \end{aligned}$$

The probability of an error is then calculated, as follows.

$$\begin{aligned} \mathbf{P}(\text{error}) &\approx .5(.0668) + .5(.0062) \\ \mathbf{P}(\text{error}) &\approx \boxed{.0365} \end{aligned}$$

4.

(a) We obtain the marginal pdf by integrating the joint pdf. Hence, we have
$f_Y(y) = \int_0^{1-y} f_{X,Y}(x,y)dx = \begin{cases} 3(1-y)^2, & 0 < y \leq 1 \\ 0, & \text{otherwise} \end{cases}$

(b) Using the answer from part (a), we obtain
$f_{X|Y}(x|0.5) = \frac{f_{X,Y}(x,y)}{f_Y(0.5)} = \begin{cases} 8x & 0 < x \leq 0.5 \\ 0, & \text{otherwise} \end{cases}$
Then, we just use the definitions of expectation and variance to obtain
$\mathbf{E}[X|Y = .5] = \int_0^{0.5} x f_{X|Y}(x|0.5)dx = \frac{1}{3}$
$\sigma^2_{X|Y=.5} = \int_0^{0.5}(x - \frac{1}{3})^2 f_{X|Y}(x|0.5)dx = \frac{1}{72}$

5.

(a) Using the total expectation theorem, we obtain
$\mathbf{E}[X] = \mathbf{E}[X|A]\mathbf{P}(A) + \mathbf{E}[X|B]\mathbf{P}(B) = 1 * \frac{1}{2} + \frac{1}{3} * \frac{1}{2} = \frac{2}{3}$

(b) Using the total probability theorem, we obtain
$\mathbf{P}(D) = \mathbf{P}(D|A)\mathbf{P}(A) + \mathbf{P}(D|B)\mathbf{P}(B) = \frac{1}{2}e^{-\tau} + \frac{1}{2}e^{-3\tau}$

(c) Using the Bayes' theorem, we obtain
$\mathbf{P}(T_{1A}|D) = \frac{\mathbf{P}(D|T_{1A})\mathbf{P}(T_{1A})}{\mathbf{P}(D)} = \frac{1}{1 + e^{-2\tau}}$

(d) Using the total expectation theorem, we obtain
$\mathbf{E}[\text{Total Time Till Failure} \mid D]$
$= \tau + \mathbf{E}[\text{Time to failure after } \tau \mid D, A]\mathbf{P}(A|D) + \mathbf{E}[\text{Time to failure after } \tau \mid D, B]\mathbf{P}(B|D)$
$= \tau + \frac{1}{1+e^{-2\tau}} + (\frac{1}{3})\frac{e^{-2\tau}}{1+e^{-2\tau}}$