

LECTURE 21
Limit theorems – I

- **Readings:** Sections 7.1-7.3

- X_1, \dots, X_n i.i.d.

$$M_n = \frac{X_1 + \dots + X_n}{n}$$

What happens as $n \rightarrow \infty$?

- Why bother?
- A tool: Chebyshev's inequality
- Convergence "in probability"
- Convergence of M_n

Deterministic limits (review)

- Sequence a_n
Number a
- a_n converges to a

$$\lim_{n \rightarrow \infty} a_n = a$$

" a_n eventually gets and stays (arbitrarily) close to a "

- For every $\epsilon > 0$,
there exists n_0 ,
such that for all $n \geq n_0$,
we have $|a_n - a| \leq \epsilon$.

Chebyshev's inequality

- Random variable X

$$\sigma^2 = \int (x - \mathbf{E}[X])^2 f_X(x) dx$$

$$\sigma^2 \geq c^2 \mathbf{P}(|X - \mathbf{E}[X]| \geq c)$$

$$\mathbf{P}(|X - \mathbf{E}[X]| \geq c) \leq \frac{\sigma^2}{c^2}$$

$$\mathbf{P}(|X - \mathbf{E}[X]| \geq k\sigma) \leq \frac{1}{k^2}$$

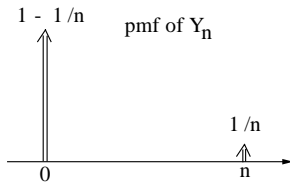
Convergence "in probability"

- Sequence of random variables Y_n
- converges in probability to a number a :
"(almost all) of the PMF/PDF of Y_n ,
eventually gets concentrated
(arbitrarily) close to a "

- For every $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbf{P}(|Y_n - a| \geq \epsilon) = 0$$

Examples



Does Y_n converge?

- Flip fair coin n times
 $Y_n = (\text{number of heads}) - n/2$
 Converges?

Convergence of the sample mean

(Weak law of large numbers)

- X_1, X_2, \dots i.i.d.
 finite mean μ and variance σ^2

$$M_n = \frac{X_1 + \dots + X_n}{n}$$

- $E[M_n] =$

- $\text{Var}(M_n) =$

$$P(|M_n - \mu| \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2}$$

The pollster's problem

- f : fraction of population that do XYZ
- i th person polled:

$$X_i = \begin{cases} 1, & \text{if yes,} \\ 0, & \text{if no.} \end{cases}$$

- $M_n = (X_1 + \dots + X_n)/n$
 fraction of "yes" in our sample

- Suppose we want:

$$P(|M_n - f| \geq .01) \leq .05$$

- Use Chebyshev's inequality:

$$\begin{aligned} P(|M_n - f| \geq .01) &\leq \frac{\sigma_{M_n}^2}{(0.01)^2} \\ &= \frac{\sigma_x^2}{n(0.01)^2} \leq \frac{1}{4n(0.01)^2} \end{aligned}$$

- If $n = 50,000$,
 then $P(|M_n - f| \geq .01) \leq .05$
 (conservative)