Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2005)

Problem Set 7G
Due: April 6, 2005

G0. Omit Questions 3 and 4 in Problem Set 7.

G1. Consider a pair (X_1, X_2) of jointly Gaussian random variables:

$$f_{(X_1,X_2)}(x_1, x_2) = \frac{1}{2\pi\sqrt{|K|}} e^{-(\underline{x}-\underline{\mu})^{\mathrm{t}} K^{-1}(\underline{x}-\underline{\mu})/2},$$

where K is a symmetric positive definite matrix, $\underline{x}$ is the column vector $(x_1, x_2)^{\mathrm{t}}$, and $\underline{\mu}$ is a two dimensional column vector $(\mu_1, \mu_2)^{\mathrm{t}}$. ($|K|$ denotes the determinant of the matrix K, and K^{-1} is the inverse matrix of K).

(a) Prove that there are scalars a, b, c and d such that:

$$\begin{aligned} X_1 &= aU + bV \\ X_2 &= cU + dV, \end{aligned}$$

where U and V are two independent normal random variables. Consequently X_1 and X_2 have a bivariate normal distribution (as defined in the notes in section 4.7)

(b) Let $Y = \alpha X_1 + \beta X_2 + \gamma$. What is the probability law of Y?

(c) Find the multivariate transform of (X_1, X_2).

G2. In general, linear estimators based on multiple measurements are not difficult to derive, but the algebra (and calculus) can become quite tedious. For this problem, derive the linear least squares estimator of X when two measurements are available. More specifically, find a_1, a_2, and b such that $g(Y_1, Y_2) = a_1 Y_1 + a_2 Y_2 + b$ minimizes

$$\mathbf{E}[(X - g(Y_1, Y_2))^2] = \mathbf{E}[(X - a_1 Y_1 - a_2 Y_2 - b)^2]$$

To make the problem a little more tractable, assume that Y_1 and Y_2 are uncorrelated (i.e., $\mathbf{E}[Y_1 Y_2] = \mathbf{E}[Y_1]\mathbf{E}[Y_2]$).