In-Class Problems — Week 9, Mon

Problem 1. Theory Hippo recently lost a lot of money playing poker with the 6.042 staff. So, Theory Hippo has decided to spend a week in Las Vegas studying gambling. The hotel manager told Theory Hippo that a full house (three-of-a-kind and one pair) is much more valuable than two pairs. Since Theory Hippo is not so sure, he decides to count the number of hands with two pairs, but no three- or four-of-a-kind. In other words, the hand $\langle 9\clubsuit, 9\diamondsuit, 8\clubsuit, 8\heartsuit, 7\spadesuit\rangle$ is considered two pairs, but $\langle 9\clubsuit, 9\diamondsuit, 8\clubsuit, 8\heartsuit, 9\heartsuit\rangle$ and $\langle 9\clubsuit, 9\diamondsuit, 9\heartsuit, 9\spadesuit, 7\clubsuit\rangle$ are not.

Theory Hippo reasons that there are 13 choices for the value of the cards in the first pair. Then, there are $\binom{4}{2} = 6$ ways to choose 2 of the 4 cards in the deck that have this value. That leaves 12 choices for the value of the cards in the second pair, and $\binom{4}{2} = 6$ ways to choose 2 of the 4 cards with this value. Finally, the fifth card can be any one of the 44 cards remaining in the deck, because the fifth card can not have the same value as the first two pairs leaving $48 - 4$ possibilities. Altogether there are $13 \cdot 6 \cdot 12 \cdot 6 \cdot 44 = 247,104$ hands with just two pairs.

Is Theory Hippo's argument correct? If so, prove it by arguing that everything has been counted exactly once. If not, explain why and how to fix it.

Problem 2. One algorithm for checking whether an arbitrary labeled graph is 3-colorable is to enumerate all possible 3-color assignments for the vertices of the graph and then test each color assignment to see if it is a valid coloring. We are going to count the maximum number of color assignments that this algorithm may have to test. (In other words, what is the worst-case behavior of this approach?)

Let $G = (V, E)$ be an n-vertex graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$, and let $C = \{R, G, B\}$ be the set of colors. A *color assignment* is a function $f : V \to C$. (*Note:* a *valid* coloring is one where $f(u) \neq f(v)$ for all $(u, v) \in E$, that is, no two adjacent vertices in the graph have the same color. We will not be concerned much with validity in this problem.)

(a) How many different color assignments are there? (Colors are not interchangable.)

(b) There are much easier ways to check if a graph can be colored with 1 color or 2 colors. Therefore, we are really only interested in color assignments that use *exactly* 3 colors. How many ways can we color the vertices of G, so that all three colors in C are used?

(c) For a graph with 3 vertices we can see that the number of color assignments with exactly 3 colors are just the number of possible permutations of R, G, B, which is 3!. Verify that your formula in part (b) gives the right answer for a 3-vertex graph.

(d) Even so, many of the assignments are redundant. For example,in a 4-vertex graph, the color assignment $RGBB$ is essentially the same as $RBGG$, because if one fails, the other does also. In general, any two colorings are equivalent if one assignment can be achieved from the other by just swapping color names. If we remove these redundant color assignments, how much can we improve our number from part (b)?