Massachusetts Institute of Technology
6.042J/18.062J, Fall '02: Mathematics for Computer Science
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In-Class Problems — Week 3, Wed

Problem 1. Let B be a set of numbers and let R be the divisibility relation on the set B. In other words, aRb iff $a \mid b$, that is, iff $ak = b$ for some natural number k.

(a) Let $B = \{2, 3, 4, 5, 6, 7, 8\}$. Draw a *Directed Acyclic Graph* (DAG) for the poset (B, R). What are the minimal and maximal elements? [1]

(b) Let B be the set, $\mathbb{N}^+$, of integers greater than 1. What are the minimal and maximal elements?

(c) Let B be the set, $\mathbb{N}$, of all natural numbers (including zero). What are the maximal and minimal elements?

Problem 2. For each student in our 6.042 class, we can assign a pair (s, a) where s represents the size (height) of the student and a represents the age of the student. Consider the relation, R, between pairs defined by the following condition:

$$(s_1, a_1)\, R\, (s_2, a_2) \quad \text{iff} \quad s_1 \leq s_2 \wedge a_1 \leq a_2.$$

At last count, our 6.042 class contained approximately 150 students. Prove that the class must contain either:

1. A set S of 13 students such that, if they line up according to increasing height, they are also arranged in increasing order of age, or
2. a set T of 13 students such that, if they line up according to increasing height, they are also arranged in decreasing order of age.

To eliminate some confusion: in these problems, when we say "increasing" we mean *weakly* increasing. A *weakly increasing* sequence may stay the same at some steps, it just never gets smaller; a *strictly increasing* sequence gets bigger at *every* step. For example, the sequence $\{2, 2, 3, 4\}$ is in *weakly* increasing order, but not in *strictly* increasing order.

Hint: Use:

[1] An element a is **maximal** in the poset $(S, \preceq)$ if there is no $b \in S$ such that $a \prec b$. Similarly, a is **minimal** if there is no element $b \in S$ such that $b \prec a$.

Theorem (Dilworth). *For all $t \in \mathbb{N}$, every poset with n elements must have* either *a chain of size at least t,* or *an antichain of size at least n/t.*

Problem 3. A pair of 6.042 TAs, Adrian and Min, have decided to devote some of their spare time this term to establishing dominion over the entire galaxy. Recognizing this as an ambitious project, they worked out the following table of tasks on the back of Min's copy of the lecture notes.

1. **Devise a logo** and cool imperial theme music - 8 days.
2. **Build a fleet** of Hyperwarp Stardestroyers out of eating paraphernalia swiped from Lobdell - 18 days.
3. **Seize control** of the United Nations - 9 days, after task #1.
4. **Get shots** for Adrian's cat, Emilios - 11 days, after task #1.
5. **Open a Starbucks chain** for the army to get their caffeine - 10 days, after task #3
6. **Train an army** of elite interstellar warriors by dragging people to see *The Phantom Menace* dozens of times - 4 days, after tasks #3, #4, and #5.
7. **Launch the fleet** of Stardestroyers, crush all sentient alien species, and establish a Galactic Empire - 6 days, after tasks #2 and #6.
8. **Defeat Microsoft** - 8 days, after tasks #2 and #6.

(a) Express the information in the task list using some type of graph (label the vertices to reflect task lengths).

(b) Give some valid order in which the tasks might be completed.

Adrian and Min want to complete all these tasks in the shortest possible time. However, they have agreed on some constraining work rules.

- Only one person can be assigned to a particular task; they can not work together on a single task.
- Once a person is assigned to a task, that person must work exclusively on the assignment until it is completed. So, for example, Adrian cannot work on building a fleet for a few days, run get shots for Emilios, and then return to building the fleet.

(c) Adrian and Min want to know how long conquering the galaxy will take. Min suggests dividing the total number of days of work by the number of workers, which is two. What lower bound on the time to conquer the galaxy does this give, and why might the actual time required be greater?

(d) Adrian proposes a different method for determining the duration of their project. He suggests looking at the duration of the "critical path", the most time-consuming sequence of tasks such that each depends on the one before. What lower bound does this give, and why might it also be too low?

(e) What is the minimum number of days that Adrian and Min need to conquer the galaxy? No proof is required.