Massachusetts Institute of Technology
6.042J/18.062J, Fall '02: Mathematics for Computer Science
Prof. Albert Meyer and Dr. Radhika Nagpal

Problem Set 11-12

Reading: Week 11-12 Notes

Problem 1. Prove that

(a) $\Pr\{A \mid B\} = \Pr\{A \mid \overline{B}\}$ iff A and B are independent.

(b) If A, B, C are mutually independent events, then A and $B \cup C$ are independent.

Problem 2. There is a course—not 6.042, naturally—in which *10% of the assigned problems contain errors*. If you pick a random problem and send an email to your TA and your Lecturer asking whether the problem has an error, then *the TA's reply will be correct 80% of the time*. This 80% accuracy holds regardless of whether or not a problem has an error. Likewise, *the Lecturer's reply will be correct with only 75% accuracy*.[1]

Furthermore, the TA and Lecturers tend to be confused by different kinds of problems. The net result of this is that *the correctness of the lecturers' answer and the TA's answer are independent of each other, regardless of whether there is an error*.

For the following parts, designate events as follows:

$$T ::= \text{"the TA's email says the problem has an error,"}$$
$$L ::= \text{"the Lecturer's email says the problem has an error."}$$
$$E ::= \text{"the problem has an error,"}$$

(a) Using T, L and E translate the sentences in italics into probability notation. For example, "the correctness of the lecturers' answer and the TA's answer are independent of each other," could translate into some equations involving terms such as $\Pr\{T \mid E\}$ and $\Pr\{T \cap L \mid E\}$.

(b) What is the probability that *both* the Lecturer and TA report an error?

(c) Is the event that "the TA says that there is an error", independent of the event that "the lecturer says that there is an error"? Prove it.

[1] This is consistent with the view that Lecturers are chosen for their theatrical personalities, not their mastery of the material. : -)

Problem 3. In the board game Monopoly, the number of squares that a player advances in a single turn is determined by up to three rolls of a pair of dice as follows.

- Initially, the player rolls two dice, sums the results, and advances that many squares.
- If the same number comes up on both dice—this is called *rolling a double*—then the player rolls the dice a second time, sums the results, and advances that many additional squares.
- If the player rolled on double on the second roll, then the player rolls the dice a third time, sums the results, and advances that many additional squares.
- However, if third roll was a double, then the player "goes to jail". We will treat this as though he had to reset his position to where he was just before the first roll, so the total number of squares advanced by the player in this case is zero.

Assume that the dice are fair and six-sided and that rolls are mutually independent.

We want to calculate the expected number of squares that a player advances in a single turn. Here is one approach to the problem. Lets assume that the pair of dice are *always* rolled three times, with the player simply ignoring results of any roll which was preceded by a non-double roll.

Let the random variable R_i be the sum of the two dice on the ith roll, and let I_i be an indicator for the event that the ith roll was a double.

(a) Let the random variable R, be the number of squares advanced by the player. Express R is terms of R_i and I_i.

(b) Calculate $\mathrm{E}[R]$. You will almost surely use some assumptions about independence. Be sure to indicate where you do so.

Problem 4. Suppose successive digits from zero to nine are generated independently until the four digit sequence **9999** appears. What is the expected number of digits generated? (*Hint:* Parse the sequence into consecutive "tries to get 9999", where a "try" is either a single digit other than 9, or a 9 followed by a single digit other than 9, etc. For example, the sequence 92349982999769999 parses into eight tries: $/92/3/4/998/2/9997/6/9999/$.)

Problem 5. Suppose that I choose a permutation of the numbers $1, 2, \ldots, n$ uniformly at random. What is the expected number of entries that are greater than all preceding entries? For example, in the permutation 4, 2, 1, 5, 3, the numbers 4 and 5 are greater than all preceding entries (Hint: What's the probability that the first entry is greater than all the preceding entries? What about the second one?)

Problem 6. There are n MIT students who are taking 6.042 and 6.003 this term. To make it easier on themselves, the professors in charge of these classes have decided to randomly permute their class lists and then assign students grades based on their rank in the permutation[2]. Assume all permutations are equally likely and that the ranking in each class is independent of the other.

(a) What is the expected number of students that have a higher rank in 6.042 than 6.003?

(b) What is the expected number of students that have a ranking at least k higher in 6.042 than in 6.003?

Problem 7. **(a)** Suppose we flip a fair coin until two heads in a row come up. What is the expected number, F, of flips we perform?

Hint: Let F_{T} be the expected number of further flips until two heads comes up, given that the previous flip was T, and likewise let F_{H} be the expected number of further flips until two heads comes up given that the previous flip was H. Argue that F_{T} will equal 1 plus the average of F_{T} and F_{H}.

(b) Suppose we flip a fair coin until a head followed by a tail come up. What is the expected number, G, of flips we perform?

(c) Suppose we now play a game: flip a fair coin until either HH or HT first occurs. You win if HT comes up first, lose if HH comes up first. What odds should you offer an opponent to make this a fair game?

[2] . . . just as many students have long suspected : -)