

Deviation from the Mean

1 What the Mean Means

We have focused on the expectation of a random variable because it indicates the "average value" the random variable will take. But what precisely does this mean?

We know a random variable may never actually equal its expectation. We also know, for example, that if we flip a fair coin 100 times, the chance that we actually flip *exactly* 50 heads is only about 8%. In fact, it gets less and less likely as we continue flipping that the number of heads will exactly equal the expected number, *e.g.*, the chance of exactly 500 heads in 1000 flips is less than 3%, in 1,000,000 flips less than 0.1%, ...

But what is true is that the fraction of heads flipped is likely to be *close* to half of the flips, and the more flips, the closer the fraction is likely to be to 1/2. For example, the chance that the fraction of heads is within 5% of 1/2 is

- more than 24% in 10 flips,
- more than 38% in 100 flips,
- more than 56% in 200 flips, and
- more than 89% in 1000 flips.

These numbers illustrate the single most important phenomenon of probability: the average value from repeated experiments is likely to be close to the expected value of one experiment. And it gets more likely to be closer as the number of experiments increases. This result was first formulated and proved by Jacob D. Bernoulli in his book *Ars Conjectandi* (The Art of Guessing) published posthumously in 1713. In his Introduction, Bernoulli comments that[1]

> even the stupidest man—by some instinct of nature *per se* and by no previous instruction (this is truly amazing)—knows for sure that the more observations . . . that are taken, the less the danger will be of straying from the mark.

But he goes on to argue that this instinct should not be taken for granted:

[1]These quotes are taken from Grinstead & Snell, *Introduction to Probability*, American Mathematical Society, p. 310.