Massachusetts Institute of Technology
6.042J/18.062J, Fall '02: Mathematics for Computer Science
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Problem Set 3

Reading: Week 3 Notes, Rosen §6.1, 6.3–6.6 (ignore Warshall's algorithm and lattices).

Note: This problem set contains a number of questions that will require you to write proofs. The goal is not only to have correct proofs, but also to make sure they are clear, orderly, and well-presented.

Problem 1. Let $(a, b) \sim (c, d)$ if $a + d = b + c$. We claim that $\sim$ is an equivalence relation.

(a) Partition the set $\{0, 1, 2\} \times \{0, 1, 2\}$ into the distinct equivalence classes under $\sim$.

(b) Prove that $\sim$ is an equivalence relation on $\mathbb{Z} \times \mathbb{Z}$.

(c) How many equivalence classes under $\sim$ are there on the set $\{0, 1, \ldots, n\} \times \{0, 1, \ldots, n\}$. Explain.

Problem 2. Let $A = \{1, 2, \ldots, n\}$ and consider the partial order $(\mathcal{P}(A), \subseteq)$.Remember that $\mathcal{P}(A)$ is the set of all subsets of A and that $\subseteq$ is the normal subset relation.

(a) For the case when $n = 3$, draw the DAG for the relation $(\mathcal{P}(A), \subseteq)$ and label two maximal chains and two maximal antichains. Give an example of a topological sort of this relation.

(b) Prove that there exists a chain of length $n + 1$ in $(\mathcal{P}(A), \subseteq)$.

(c) Prove that for any integer k such that $0 < k < n$, the set $\{B | B \subseteq A \text{ and } |B| = k\}$ is an antichain in $(\mathcal{P}(A), \subseteq)$.

Problem 3. Let R be the lexicographic (dictionary) ordering of strings of symbols from a totally ordered alphabet as defined in Rosen p. 417.

(a) Prove that R is a total order.

(b) Prove that neither R nor R^{-1} is a well-ordering, for alphabets with more than one element. Recall that a well-ordering on a set S is a total ordering on S such that every non-empty subset of S has a smallest element.

Problem 4. Let R be a symmetric relation on a set A. Show that R^n is symmetric for each positive integer n. Deduce that R^* is symmetric. *Hint:* Use induction.

Problem 5. For any function $f : A \to B$, define a binary relation, R_f, on A by the condition:

$$a_1 R_f a_2 \longleftrightarrow f(a_1) = f(a_2).$$

(a) Show that R_f is an equivalence relation.

(b) Show that for *every* equivalence relation E on A, there exists a set B and a function $f : A \to B$ such that $E = R_f$.