Massachusetts Institute of Technology
6.042J/18.062J, Fall '02: Mathematics for Computer Science
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Solutions to In-Class Problems — Week 1, Wed

Problem 1. Generalize the proof from lecture[1] that $\sqrt{2}$ is irrational, *e.g.*, how about $\sqrt[3]{2}$?

Solution. We prove that for any $n > 1$, $\sqrt[n]{2}$ is irrational by contradiction.

Assume that $\sqrt[n]{2}$ is rational. Under this assumption, there exists integers a and b with $\sqrt[n]{2} = a/b$, where a and b have no common factors (so that the fraction a/b is in lowest terms). Now we prove that a and b are both even, which contradicts the existence of a, b.

$$\begin{aligned} \sqrt[n]{2} &= \frac{a}{b} \\ 2 &= \frac{a^n}{b^n} \\ 2b^n &= a^n. \end{aligned}$$

The lefthand side of the last equation is even, so a^n is even. This implies that a is even as well (see below for justification).

In particular, $a = 2c$ for some integer c. Thus,

$$\begin{aligned} 2b^n &= (2c)^n = 2^n c^n, \\ b^n &= 2^{n-1} c^n. \end{aligned}$$

Since $n - 1 > 0$, the righthand side of the last equation is an even number, so b^n is even. But this implies that b must be even as well, contradicting the fact that m/n is in lowest terms.

Now we justify the claim that if b^k is even, so is b.

The original argument that $\sqrt{2}$ is irrational (repeated on the last page of this handout) used the special case for $k = 2$: if the square of an integer is even, then the integer itself is even. For this case, there's a simple proof by contradiction: Suppose b was not even. Then b is odd, so b^2 would also be odd—using the easy fact that the product of two odd numbers is odd—which contradicts the fact that b^2 is even.

But more generally for *any* integer $m, k > 0$, if m^k is divisible by a prime number, p, then m must be divisible by p. This follows from the factorization of an integer into primes: the primes in the factorization of m^k are precisely the primes in the factorization of m repeated k times, so if there is a p in the factorization of m^k it must be one of k copies of a p in the factorization of m. ■

[1] See attached. This one not a false proof.

[Optional]

Here's a somewhat broader generalization of the proof that $\sqrt{2}$ is irrational.

Lemma. *Let the coefficients of the polynomial* $a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + x^n$ *be integers. Then any real root of the polynomial is either integral or irrational.*

Notice that this Lemma directly implies that $\sqrt{2}$ is irrational: $x^2 - 2$ has no integer root because 2 is not a perfect square, so the real roots, namely, $\pm\sqrt{2}$, must be irrational. Similarly, if $m \in \mathbb{N}$ is not the kth power of an integer, then $x^k - m$ has no integer roots, so $\sqrt[k]{m}$ will be irrational.

Proof. Let $r \in \mathbb{R}$ be a root of the polynomial, that is,

$$a_0 + a_1r + a_2r^2 + \cdots + a_{n-1}r^{n-1} + r^n = 0.$$

There are three cases for r: it is either integral, irrational, or the ratio of two integers with no common divisors where the denominator is greater than one. We want to eliminate the last case.

So assume to the contrary that $r = s/t$ for integers s and t which have no common divisors and such that $t > 1$. Now t must have a prime divisor, p.

Substituting s/t for r and multiplying both sides of the above equation by t^n yields:

$$a_0t^n + a_1st^{n-1} + a_2s^2t^{n-2} + \cdots + a_{n-1}s^{n-1}t + s^n = 0, \quad (1)$$

$$a_0t^n + a_1st^{n-1} + a_2s^2t^{n-2} + \ldots + s^{n-1}t = -s^n. \quad (2)$$

Then p divides each term of the lefthand side of equation (2), and therefore also divides the righthand side, viz., $p \mid -s^n$. But this means that p must also divide s. So p is a common divisor of s and t, contradicting our choice of s and t.

□

Problem 2. Albert & Radhi announce that they plan a surprise 6.042 quiz next week. Their students wonder if the quiz could be next Friday. The students realize that it obviously cannot, because if it hadn't been given before Friday, everyone would know that there was only Friday left on which to give it, so it wouldn't be a surprise any more.

So the students ask whether Albert & Radhi could give the surprise quiz Thursday? They observe that if the quiz wasn't given *before* Thursday, it would have to be given *on* the Thursday, since they already know it can't be given on Friday. But having figured that out, it wouldn't be a surprise if the quiz was on Thursday either. Similarly, the students reason that the quiz can't be on Wednesday, Tuesday, or Monday. Namely, it's impossible for Albert & Radhi to give a surprise quiz next week. All the students now relax having concluded that Albert & Radhi must have been bluffing.

And since no one expects the quiz, that's why, when Albert & Radhi give it on Tuesday next week, it really is a surprise!

What do you think is wrong with the students' reasoning? Can you fix it?

Solution. The basic problem is that "surprise" is not a mathematical concept, nor is there any sure, clear way to give it a mathematical definition. The "proof" above assumes some plausible axioms about surprise, whatever that is, and the paradox is that these axioms are inconsistent. But that's no surprise :-), since we don't know what we're talking about.

Mathematicians and philosophers have had a lot more to say about what might be wrong with the students' reasoning, (see Chow, Timothy Y. *The surprise examination or unexpected hanging paradox*, American Math. Monthly (January 1998), pp.41–51.) ■

Appendix

Theorem. $\sqrt{2}$ *is an irrational number.*

Remember that an irrational number is a number that can not be expressed as a ratio of two integers.

Proof. The proof is by contradiction. Assume for purpose of contradiction that $\sqrt{2}$ is rational.

Then we can write $\sqrt{2} = m/n$ where m and n are integers and the fraction is in lowest terms. Squaring both sides gives $2 = m^2/n^2$, so $2n^2 = m^2$. This implies that m^2 is even, and hence that m is even; that is, m is a multiple of 2. But that means m^2 is actually a multiple of 4, say $m^2 = 4k$.

Now we have $2n^2 = m^2 = 4k$, so $n^2 = 2k$. So n^2 is even, and hence n is even. But since m and n are both even, the fraction m/n is not in lowest terms, a contradiction. $\square$