Massachusetts Institute of Technology
6.042J/18.062J, Fall ’02: Mathematics for Computer Science
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Problem Set 10

Reading: For this Problem Set: Week 10 Notes, §1–9; Optional: Rosen §§4.4 & 4.5 through Example 4, p.271.

For this Week’s Lectures and Class Problems: Complete Week 10 Notes; Optional: Rosen §§4.4 & 4.5 through Example 9, p. 274.

Problem 1. There are 4 different coins in a box. The probability of Heads when flipping the ith coin is $1/i$ for $1 \leq i \leq 4$. A coin is selected from the box randomly, and gets tossed until a Head appears.

(a) Write down a probability space for the experiment. Be sure to verify that the sum of the probabilities of the sample points is 1.

(b) What is the probability that a Head is first seen in the 2nd toss?

(c) Given that a Head is first seen in the 2nd toss, what is the probability that ith coin was selected from the box?

(d) Check your answer to part (c) to be sure it satisfies:

$$\Pr\{C_1 \mid H_2\} + \Pr\{C_2 \mid H_2\} + \Pr\{C_3 \mid H_2\} + \Pr\{C_4 \mid H_2\} = 1.$$

Problem 2. We consider a variation of Monty Hall’s game. The contestant still picks one of three doors, with a prize randomly placed behind one door and goats behind the other two. But now, instead of always opening a door to reveal a goat, Monty instructs Carol to *randomly* open one of the two doors that the contestant hasn’t picked. This means she may reveal a goat, or she may reveal the prize. If she reveals the prize, then the entire game is *restarted*, that is, the prize is again randomly placed behind some door, the contestant again picks a door, and so on until Carol finally picks a door with a goat behind it. Then the contestant can choose to stick with his original choice of door or switch to the other unopened door. He wins if the prize is behind the door he finally chooses at this point; otherwise he loses.

To analyze this setup, define two events:

GP**:** The event that the contestant **g**uesses the door with the **p**rize behind it on his first guess.

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OP**:** The event that the game is restarted at least once. Another way to describe this is as the event that the door Carol first **o**pens has a **p**rize behind it.

(a) What is $\Pr\{OP\}$? ... $\Pr\{GP \mid \overline{OP}\}$?

(b) What is the probability the game will end after *exactly* $n \geq 1$ guesses by the contestant?

(c) Is it possible for the game never to end? What is the probability that the game will end?

When Carol finally picks the goat, the contestant has the choice of sticking or switching. The contestant figures that since this is a 6.042 problem with the Monty Hall rules changed to mislead him :-), it's a good guess that the better strategy in the original game will not be better in this game. So he decides to use the *sticking strategy*: when Carol gives him a chance to switch, he will stick with the door he last guessed.

(d) Describe a simple probability space modelling this game, and verify that the sum of the probabilities of the sample points is 1. The space should have enough sample points to describe the event, W_n, that the game contestant wins after making *at least* n guesses, for every $n \geq 1$. Describe the outcomes in W_n.

(e) Let $w ::= \Pr\{W\}$, where W is the event that the contestant *w*ins with the sticking strategy. Express the following probabilities as simple closed forms in terms of w.

i) $\Pr\{W \mid \overline{GP} \cap \overline{OP}\}$
ii) $\Pr\{W \mid \overline{GP} \cap OP\}$
iii) $\Pr\{W \mid GP\}$
iv) $\Pr\{W\}$

(f) For any final outcome where the contestant wins with a "stick" strategy, he would lose if he had used a "switch" strategy, and vice versa. In the original Monty Hall game, we concluded immediately that the probability that he would win with a "switch" strategy was $1 - \Pr\{W\}$. Why isn't this conclusion quite as obvious for this new, restartable game? Is this conclusion still sound? Briefly explain.

Problem 3. **(a)** Suppose that you are looking in your desk for a letter from some time ago. Your desk has eight drawers, and you assess the probability that it is in any particular drawer as 10% (so there is a 20% chance that it is not in the desk at all). Suppose now that you start searching systematically through your desk, one drawer at a time. In addition, suppose that you have not found the letter in the first i drawers, where $0 \leq i \leq 7$. Let p_i denote the probability that the letter will be found in the next drawer, and let q_i denote the probability that the letter will be found in some subsequent drawer (both p_i and q_i are conditional probabilities, since they are based on the assumption that the letter is not in the first i drawers).

Find formulas for p_i and q_i, and conclude that p_i is a strictly increasing function of i, and q_i is strictly decreasing.

Hint: Observe that if event A implies event B then $\Pr\{A \mid B\} = \Pr\{A\} / \Pr\{B\}$

(b) The following data appeared in an article in the Wall Street Journal. For the ages 20, 30, 40, 50, and 60, the probability of a woman in the U.S. developing cancer in the next ten years is 0.5%, 1.2%, 3.2%, 6.4%, and 10.8%, respectively. At the same set of ages, the probability of a woman in the U.S. eventually developing cancer is 39.6%, 39.5%, 39.1%, 37.5%, and 34.2%, respectively. This seems strange, but use the previous part of the problem to give an explanation for these data.