Permutations and Combinations

In Notes 8, we saw a variety of techniques for counting elements in a finite set: the Sum Rule, Inclusion-Exclusion, the Product Rule, tree diagrams, and permutations. We will now introduce yet another rule, the *Division Rule*, and one more concept, *combinations*. We will also learn techniques for counting elements of a finite set when limited repetition is allowed.

1 The Division Rule

The division rule is a common way to ignore "unimportant" differences when you are counting things. You can count distinct objects, and then use the division rule to "merge" the ones that are not significantly different.

We will state the Division Rule twice, once informally and then again with more precise notation.

Theorem 1.1 (Division Rule). *If B is a finite set and $f : A \mapsto B$ maps precisely k items of A to every item of B, then A has k times as many items as B.*

For example, suppose A is a set of students, B is a set of tutorials, and f defines the assignment of students to tutorials. If 12 students are assigned to every tutorial, then the Division Rule says that there are 12 times as many students as tutorials.

The following two definitions permit a more precise statement of the Division Rule.

Definition 1.2. If $f : A \mapsto B$ is a function, then $f^{-1}(b) = \{a \in A \mid f(a) = b\}$.

That is, $f^{-1}(b)$ is the set of items in A that are mapped to the item $b \in B$. In the preceding example, $f^{-1}(b)$ is the set of students assigned to tutorial b.

This notation can be confusing, since f^{-1} normally denotes the inverse of the function f. With our definition, however, $f^{-1}(b)$ can be a set, not just a single value. For example, if f assigns no items in A to some element $b \in B$, then $f^{-1}(b)$ is the empty set. In the special case where f is a bijection, $f^{-1}(b)$ is always a single value, and so f^{-1} by our definition is just the ordinary inverse of f.

Definition 1.3. A function $f : A \mapsto B$ is *k-to-1* if for all $b \in B$, $|f^{-1}(b)| = k$.

For example, if f assigns exactly 12 students to each recitation, then f is 12-to-1. Assuming k is non-zero, a k-to-1 function is always a surjection; every element of the range is mapped to by $k > 0$ elements of the domain.

We can now restate the Division Rule more precisely:

Theorem (Division Rule, restatement). *If B is a finite set and $f : A \mapsto B$ is k-to-1, then $|A| = k|B|$.*