

6	9	13	7
12	10	5	
3	4	14	
15	8	11	2

Laws of Probability

6	9	13	7
12	10	5	
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15	8	11	2

Probability Spaces

- 1) Sample space, \mathcal{S} , whose elements are called **outcomes**.
- 2) Probability function,
 $\text{Pr}: \mathcal{S} \rightarrow [0,1]$
 - (a) $\text{Pr}\{\mathcal{S}\} = 1$,
 - (b) the **Sum Rule**:

6	9	13	7
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Basic Laws of Probability

Sum Rule

$$\begin{aligned} \text{Pr}\{A_1 \cup A_2\} \\ = \text{Pr}\{A_1\} + \text{Pr}\{A_2\} \\ \text{for } A_1 \cap A_2 = \emptyset. \end{aligned}$$

6	9	13	7
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Sum Rule for Sets

Corresponding Rule for Sets:

$$\begin{aligned} |A_1 \cup A_2| &= |A_1| + |A_2| \\ \text{for } A_1 \cap A_2 &= \emptyset. \end{aligned}$$

6	9	13	7
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Difference Rule

$$\begin{aligned} \text{Pr}\{A - B\} &= \\ &\text{Pr}\{A\} - \text{Pr}\{A \cap B\} \end{aligned}$$

Proof:
 A is the disjoint union of $A - B$ and $A \cap B$.

6	9	13	7
12	10	5	
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Complement Rule

$$\text{Pr}\{\bar{B}\} = 1 - \text{Pr}\{B\}$$

Proof:

$$\begin{aligned} \text{Pr}\{\bar{B}\} &= \text{Pr}\{\mathcal{S} - B\} \\ &= \text{Pr}\{\mathcal{S}\} - \text{Pr}\{\mathcal{S} \cap B\} \text{ (Diff Rule)} \\ &= 1 - \text{Pr}\{B\}. \end{aligned}$$

6	9	13	7
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Inclusion-Exclusion

$$\Pr\{A \cup B\} = \Pr\{A\} + \Pr\{B\} - \Pr\{A \cap B\}$$

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L10-2.7

6	9	13	7
12	10	5	
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Boole's Inequality

$$\Pr\{A \cup B\} \leq \Pr\{A\} + \Pr\{B\}$$

Monotonicity

$$\Pr\{A\} \leq \Pr\{A \cup B\}$$

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L10-2.8

6	9	13	7
12	10	5	
3	4	14	
15	8	11	2

Basic Laws of Probability

Infinite Sum Rule

$$\Pr\{A_1 \cup A_2 \cup \dots\} = \Pr\{A_1\} + \Pr\{A_2\} + \dots$$

for **pairwise disjoint** A_n .

$$\Pr\{\cup_{n \in N} A_n\} = \sum_{n \in N} \Pr\{A_n\}$$

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L10-2.9

6	9	13	7
12	10	5	
3	4	14	
15	8	11	2

Boole's Inequality

$$\Pr\{\cup_{n \in N} A_n\} \leq \sum_{n \in N} \Pr\{A_n\}$$

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L10-2.10

6	9	13	7
12	10	5	
3	4	14	
15	8	11	2

Boole's Inequality

Example: 10,000 transistor chip with $\Pr\{\text{transistor failure}\} \approx 1/1,000,000$.
Chip fails if any transistor fails.

$$\begin{aligned} \Pr\{\text{Chip fails}\} &= \Pr\{\cup [i\text{th transistor fails}]\} \\ &\leq \sum \Pr\{i\text{th transistor fails}\} \\ &\approx (1/M) \cdot (10,000) = 0.01 \end{aligned}$$

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L10-2.11

6	9	13	7
12	10	5	
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Boole's Inequality

So $\Pr\{\text{Chip works}\} \approx 0.99$,
without any assumptions on relations between transistor failures, e.g., whether all chips fail together, or never fail together, or are unrelated.

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L10-2.12

6	9	13	7
12	10	5	
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15	8	11	2

In-class Problem

Problem 1

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L10-2.13

6	9	13	7
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Probability Analysis Method

1. Identify outcomes (*tree helps*)
2. Identify event (*e.g. winning*)
3. Assign outcome probabilities
4. Compute event probabilities (*sum outcome probabilities*)

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L10-2.14

6	9	13	7
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Assigning Outcome Probabilities

Often reason from **symmetry**:

- one 5-card hand as likely as another,
- Heads as likely as Tails,
- $\Pr\{\text{hit radius } r \text{ target}\} = (1/4) \cdot \Pr\{\text{hit radius } 2r \text{ target}\}$
- Monty Hall tree structure

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L10-2.15

6	9	13	7
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Assigning Outcome Probabilities

Also reason from **data**:

- $\Pr\{\text{family pet is a cat}\} \approx 0.69$
- $\Pr\{\text{Massachusetts voter is Republican}\} \approx 0.43$

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L10-2.16

6	9	13	7
12	10	5	
3	4	14	
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Assigning Outcome Probabilities

But probability assignment is an **axiom**. Nothing is *mathematically* wrong with bad choices, eg, uniform Monty Hall outcomes, or very skewed probabilities for Heads versus Tails. Conclusions based on bad assignments will be *perfectly consistent*, they just **won't predict reality** well.

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L10-2.17

6	9	13	7
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Penny Flipping Experiment

- Balance 40 pennies **on their edges**. (**Separately!**)
- Pound the table till all fall.
- Record $h ::= \#\text{heads}$, $t ::= \#\text{pennies}$.
- Repeat at least 3 times.

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L10-2.18

6	9	13	7
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3	4	14	
15	8	11	2

The End

Lecture ended here without getting to subsequent slides.

6	9	13	7
12	10	5	
3	4	14	
15	8	11	2

Conditional Probability

“Knowledge” changes probabilities:

$$\Pr\{\text{die rolled } 1\} = 1/|\{1,2,3,4,5,6\}| = 1/6.$$

$$\Pr\{\text{die rolled } 1 \text{ knowing that die rolled odd number}\} = 1/|\{1,3,5\}| = 1/3.$$

6	9	13	7
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Conditional Probability

$\Pr\{A | B\} ::=$ probability of event A given that event B has occurred. Formally,

$$\Pr\{A | B\} ::= \frac{\Pr\{A \cap B\}}{\Pr\{B\}}$$

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12	10	5	
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15	8	11	2

Conditional probability for Monty Hall

$$\Pr\{\text{prize at Door 1} | \text{Carol opens 2}\} = 1/2. \text{ Really!}$$

$$[\text{Carol opens 2}] = \underbrace{\{(1,1,2), (1,3,2)\}}_{\Pr = \frac{1}{18}}, \underbrace{\{(3,3,2), (3,1,2)\}}_{\Pr = \frac{1}{9}}$$

6	9	13	7
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Conditional probability for Monty Hall

This suggests the contestant may as well **stick**, since the probability is $1/2$ given what he knows when he gets to stick or switch?

Not so: Contestant knows *more* than door opened by Carol -- also knows which door he chose himself!

6	9	13	7
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Conditional probability for Monty Hall

$$\Pr\{\text{prize at Door 1} | \text{Contestant chose 1 \& Carol opens 2}\} = 1/3.$$

$$[\text{Contestant chose 1 \& Carol opens 2}] = \underbrace{\{(1,1,2)\}}_{\Pr = \frac{1}{18}}, \underbrace{\{(3,1,2)\}}_{\Pr = \frac{1}{9}}$$



6	12	7
12	12	8
7	8	14
14	8	8

Product Rule

$$\Pr\{A \cap B\} \\ = \Pr\{A \mid B\} \Pr\{B\}$$



6	12	7
12	12	8
7	8	14
14	8	8

In-class Problem

Problems 2&3