Massachusetts Institute of Technology
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In-Class Problems — Week 14, Wed

Problem 1. A gambler has been studying the roulette wheel in a Las Vegas casino over a long period of time, and his data confirms that the wheel is biased to come up on a red number about 51% of the time. Since a bet on red pays even money (bet $1 and either lose it, or win and get back $2), he realizes the odds are in his favor.

So with high hopes and an initial stake of $10,000, the gambler aims to turn his stake into $1,000,000. He plans simply to continue making $100 bets on red until either he is ahead the $1,000,000, or he goes bankrupt. He considers bankruptcy very unlikely because the odds are in his favor, and the bets are small enough relative to his initial stake that he can withstand a long run of losses.

(a) Based on this description, describe a reasonable probability space to model this situation.

(b) Describe a positive constant, ϵ, such that, at any point during his play when the gambler has not gone bankrupt, there is a probability of at least ϵ that he will reach his goal of $1,000,000 within the next 9,999 bets.

(c) Conclude that the probability that the gambler bets $n \geq 9,999$ or more times before the game ends is at most r^{n-9999} for some constant $r < 1$.

NOTE: We didn't get to any of the following three problems in class.

Problem 2. The "high hopes" of the Gambler in Problem 1 overlook some difficulties with his strategy. Is he really very likely to win? How long will it take?

Problem 3. Let G be the amount won by the gambler when a Gambler's Ruin game ends. Let Q be the number of bets till the game ends.

The derivation of the formula for the expected number of bets, $\mathrm{E}[Q]$, in an unfair Gambler's Ruin game used the fact that

$$\mathrm{E}[Q] \cdot \mathrm{E}[\text{amount won win per bet}] = \mathrm{E}[G]. \quad (1)$$

Prove this equation. *Hint:* Since the amount won per bet may be negative, Wald's Theorem does not immediately apply.

Problem 4. Prove that in an *unbounded* fair game, where the Gambler plays until he is broke no matter how much his stake increases in the meantime, the Gambler is *sure* to go broke, but the expected number of bets before he goes broke is infinite.

A Appendix

The *expectation* of random variable, R, is:

$$\mathrm{E}[R] ::= \sum_{r \in \text{range}(R)} r \cdot \Pr\{R = r\}.$$

If R has codomain $\mathbb{N}$, then this definition can also be written as

$$\mathrm{E}[R] = \sum_{r \in \mathbb{N}} \Pr\{R > r\}$$

The *conditional expectation*, $\mathrm{E}[R \mid A]$, of a random variable, R, given event, A, is

$$\mathrm{E}[R \mid A] ::= \sum_{r} r \cdot \Pr\{R = r \mid A\}. \tag{2}$$

Theorem (Law of Total Expectation). *If the sample space is the disjoint union of events $A_1, A_2, \cdots$, then*

$$\mathrm{E}[R] = \sum_{i} \mathrm{E}[R \mid A_i] \Pr\{A_i\}.$$

Theorem (Wald). *Let $C_1, C_2, \ldots,$ be a sequence of nonnegative random variables, and let Q be a positive integer-valued random variable, all with finite expectations. Suppose that*

$$\mathrm{E}[C_i \mid Q \geq i] = \mu$$

for some $\mu \in \mathbb{R}$ and for all $i \geq 1$. Then

$$\mathrm{E}[C_1 + C_2 + \cdots + C_Q] = \mu\, \mathrm{E}[Q].$$

Theorem. *In the Gambler's Ruin game with probability p of winning each individual bet, with initial capital, n, and goal, T,*

$$\Pr\{\textit{the gambler is a winner in the fair game}\} = \frac{n}{T}, \tag{3}$$

$$\Pr\{\textit{the gambler is a winner a biased game}\} = \frac{(q/p)^n - 1}{(q/p)^T - 1}. \tag{4}$$

$$\Pr\{\textit{the gambler is a winner in an unfair game}\} \leq (p/q)^{T-n}. \tag{5}$$

Let Q be the number of bets till the game ends.

$$\mathrm{E}[Q \textit{ in an unfair game}] = \frac{\Pr\{\textit{gambler is a winner}\}\, T - n}{2p - 1}.$$

$$\mathrm{E}[Q \textit{ in a fair game}] = n(T - n). \tag{6}$$