Massachusetts Institute of Technology
6.042J/18.062J, Fall '02: Mathematics for Computer Science
Professor Albert Meyer and Dr. Radhika Nagpal

In-Class Problems — Week 5, Fri

Problem 1. Consider the following game for two players. The players alternate moves. A move consists of a pair (x, y) of natural numbers, subject to the constraint that none of the previous moves may be $\preceq_c$ than the current move, where $\preceq_c$ is the coordinatewise partial order. An equivalent way to say this is that the current move must be $\prec_c$ or incomparable to all the previous moves. A player who moves to the origin $(0, 0)$ is the loser.

For example, the Player 1 might choose (5,6), after which Player 2 can move to any point (n, m) such that $n < 5$ or $m < 6$, for example, (4,12). Now the players might move successsively to (4,11), (29,5), (1,1), (0,54), (0,1). At this point it's Player 2's turn, and he can move to to (1,0). At this point it is Player 1's move, and the only available move is to the origin (0,0), so Player 1 loses this play of the game.

(a) Identify a winning strategy for the first player, and argue its correctness. Does your strategy guarantee any bound on the number of game moves?

(b) Is there any strategy that guarantees a bound on the number of game moves?

(c) Prove that the game must always terminate, even if the players conspire to choose strategies that try to keep the game going indefinitely.

NOTE October 4, 2002, 6PM: We didn't get to either of the following two problems.

Problem 2. In Week 5 Notes we considered win-lose games, ignoring games which can end in a draw. Such games have finite-path game trees in which the leaves are labelled with *win*, *lose*, or *draw*, indicating the outcome for the first player.

While a winning stategy for a player ensures the player will win no matter what moves the other player makes, there is now the possibility of a *non-losing* stategy which ensures that the player will *win or draw* no matter what moves the other player makes. (A winning strategy counts as a kind of a non-losing strategy.)

For example, in tic-tac-toe there is no winning strategy for either player, but *both* players have non-losing strategies. For chess, no one knows if white or black has a winning strategy. However, on general principles, we can be sure that at least one player has a non-losing chess strategy.

Explain why for any two-person win-lose-draw game with a finite path game tree, either one player has a winning strategy or *both* players have non-losing strategies.

Problem 3. Prove part 2. of Lemma 3.4 below.

A From Week 5 Notes

Definition 3.1. A poset $(P, \preceq)$ is *well-founded* iff every nonempty subset $S \subseteq P$ has a *minimal element*.

Lemma 3.2. *A poset is well-founded iff it has no infinite decreasing chain.*

Definition 3.3. Let $(P_1, \preceq_1)$ and $(P_2, \preceq_2)$ be partial orders. The *lexicographic partial order*, $\preceq_{\text{lex}}$, on $P_1 \times P_2$ is defined by the condition that

$$(p_1, p_2) \preceq_{\text{lex}} (q_1, q_2) \text{ iff } [p_1 \prec_1 q_1 \text{ or } (p_1 = q_1 \wedge p_2 \preceq_2 q_2)]. \tag{1}$$

The *coordinatewise partial order*, $\preceq_{\text{c}}$, on $P_1 \times P_2$ is defined by the condition that

$$(p_1, p_2) \preceq_{\text{c}} (q_1, q_2) \text{ iff } [p_1 \preceq_1 q_1 \wedge p_2 \preceq_2 q_2]. \tag{2}$$

Lemma 3.4. *Suppose $(P_1, \preceq_1)$ and $(P_2, \preceq_2)$ are posets. Then*

1. *so are $(P_1 \times P_2, \preceq_{lex})$ and $(P_1 \times P_2, \preceq_c)$. Moreover,*
2. *if $(P_1, \preceq_1)$ and $(P_2, \preceq_2)$ are both well-founded, then so are $(P_1 \times P_2, \preceq_{lex})$ and $(P_1 \times P_2, \preceq_c)$.*
3. *if $(P_1, \preceq_1)$ and $(P_2, \preceq_2)$ are both totally ordered, then so is $(P_1 \times P_2, \preceq_{lex})$.*