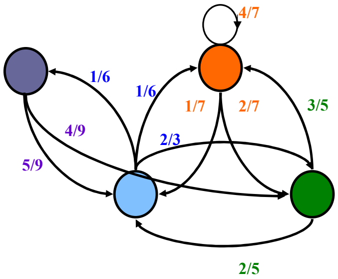
Mathematics for Computer Science

MIT 6.042J/18.062J

Random Walks

Graph with probable transitions



Graph with probable transitions

Questions

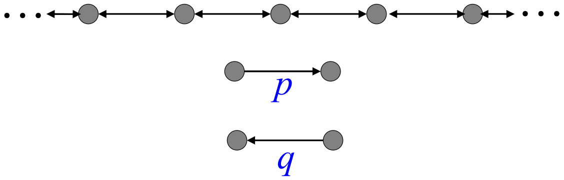
- Pr{blue reaches orange *before* green}
- Pr{blue *ever* reaches orange}
- E[#steps blue to orange]
- Average fraction of time at blue

Random Walks

Applications

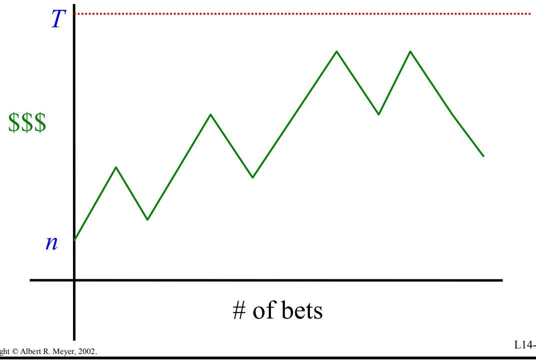
- Finance – Stocks, options
- Algorithms – web search, clustering
- Physics – Brownian Motion

1-Dimensional Walk



Gambler’s Ruin

Gambler’s Ruin



The Gambler’s Ruin

Parameters:

n ::= initial capital (stake)

T ::= gambler’s Target

p ::= Pr{win \$1 bet}

$q ::= 1 - p$

m ::= intended profit = $T - n$

The Gambler’s Ruin

Three general cases:

- Biased against $p < 1/2$
- Biased in favor $p > 1/2$
- Unbiased (Fair) $p = 1/2$

Unbiased Case: $p = q = 1/2$

Let w ::= Pr{reach Target}

$$E[\$\$] = w\cdot(T-n) + (1-w)\cdot(-n)$$

$$= wT - n$$

But game is *fair*, so E[\$\$ won] = 0

$$w = \frac{n}{T}$$

Unbiased Case

Consequences

n=500, T=600

Pr{win \$100} = 500/600 ≈ 0.83

n=1,000,000, T=1,000,100

Pr{win \$100} ≈ 0.9999

Unbiased Case -- More analysis

Wait! Why is

E[\$\$ won] = 0?

Define Random Variables, G_i

$$G_i ::= \begin{cases} 0 & \text{if game ends in } < i \text{ bets} \\ 1 & \text{if gambler wins } i^{\text{th}} \text{ bet} \\ -1 & \text{if gambler loses } i^{\text{th}} \text{ bet} \end{cases}$$

Infinite Additivity of Expectation

$$\$\$ \text{ won} = \sum_{i=1}^{\infty} G_i$$

$$\begin{aligned} E[\$\$ \text{ won}] &= E[\textstyle\sum G_i] \\ &= \textstyle\sum E[G_i] \\ &= \textstyle\sum 0 \\ &= 0 \end{aligned}$$

Infinite Additivity?

WAIT!

ALARM!

This is just like
the bet-doubling (St. Petersburg) paradox
(a fair game with guaranteed \$10 win)

Infinite Additivity?

We must verify that

$$\sum_{i=1}^{\infty} \mathrm{E}\left[\left|G_i\right|\right]$$

converges.

Infinite Additivity?

$$|G_i| = \begin{cases} 1 & \text{if} \geq i \text{ bets are made,} \\ 0 & \text{game ends before } i \text{ bets.} \end{cases}$$

$$\mathrm{E}[|G_i|] = \Pr\{\geq i \text{ bets}\}$$

Convergence Condition is Met

In-class Problem 1:

$\Pr\{\text{game takes} \geq i \text{ bets}\} \leq cr^i$

for some $c > 0$, $r < 1$, so

$$\sum_{i=1}^{\infty} \mathrm{E}\,[|G_i|] \leq c \sum_{i=1}^{\infty} r^i < \infty$$

The End is Certain

$\Pr\{\text{game takes} \geq i \text{ bets}\} \leq cr^i$

so

$\Pr\{\text{game takes forever}\} = 0.$

Already was assumed in:

$\Pr\{\text{loss}\} = 1 - w = 1 - \Pr\{\text{win}\}$

In-Class Problem

Problem 1

Biased Against: $p < 1/2 < q$

Betting red in US roulette

$$p = 18/38 = 9/19 < 1/2$$

Biased Against: $p < 1/2 < q$

Astonishing Fact!

Pr{win $100 starting with $500}

< 1/37,000 !

(was 5/6 in the unbiased case.)

Biased Against: $p < 1/2 < q$

More amazing still!

Pr{win $100 starting with $1M}

< 1/37,000

Pr{win $100 starting w/ any $*n* stake}

< 1/37,000

End Of Class Presentation

Lecture ended here on Wednesday, Dec. 4, 2002

Winning in the Biased Case

$w_n ::= \Pr\{\text{win with stake } n\}$

$$w_n = pw_{n+1} + qw_{n-1}$$

$w_0 = 0$ (*Gambler starts broke*)

$w_T = 1$ (*Gambler starts a winner*)

$$w_{n+1} = (1/p)w_n - (q/p)w_{n-1}$$

Winning in the Biased Case

$$w_{n+1} - (1/p)w_n + (q/p)w_{n-1} = 0$$

A linear recurrence: Guess that

$w_n = c^n$ for some c, so

$$c^{n+1} - (1/p)c^n - (q/p)c^{n-1} = 0$$

Winning in the Biased Case

$$c^2 - (1/p)c - (q/p) = 0$$

roots = 1, q/p so

$w_n = (q/p)^n$ and $w_n = 1^n$ satisfy

$$w_{n+1} - (1/p)w_n + (q/p)w_{n-1} = 0$$

Winning in the Biased Case

so $a\,(q/p)^n + b1^n$

satisfies the recurrence. Use boundary conditions at $n = 0,T$ to solve for a and b, and get:

Winning in the Biased Case

$$w_n = \frac{(q/p)^n - 1}{(q/p)^T - 1}$$

for $p \neq q$

Winning in the Unfair Case

Punchline: for $p < q$:

$$w_n \leq \frac{(q/p)^n}{(q/p)^T} = \left(\frac{p}{q}\right)^m$$

where $m ::= T-n$ = intended profit

Winning in the Unfair Case

for $p < q$:

$$\left(\frac{p}{q}\right)^m$$

is exponentially decreasing in m, the intended profit.

Losing in Roulette

$p = 18/38$, $q = 20/38$

$$\Pr\{\text{win } \$100\} = \left(\frac{18/38}{20/38}\right)^{100} = \left(\frac{9}{10}\right)^{100} < \frac{1}{37,648}$$

Losing in Roulette

$$\Pr\{\text{win } \$200\} = (\Pr\{\text{win } \$100\})^2$$
$$= \left(\frac{1}{37,648}\right)^2$$
$$< \frac{1}{70,000,000}$$

How Many Bets?

What is the expected number of bets for the game to end?

– either by winning $\$(T\text{-}n)$ or by going broke (losing $\$n$).

How Many Bets? Biased Case

E[\$ per bet] = $p - q = 2p - 1$

so by Wald's Thm

E[\$ won] = $(2p\text{-}1)$ E[# bets]

$$\text{E[\# bets]} = \frac{\text{E[\$ won]}}{(2p\text{-}1)}$$

How Many Bets? Biased Case

But

E[\$ won] = $w_n(T-n) - (1-w_n)n$

so

$$\boxed{\text{E[\# bets]} = \frac{w_n T - n}{2p - 1}}$$

for $p \neq 1/2$.

How Many Bets? Fair Case

E[# bets] = $n(T-n)$ =

(initial stake)·(intended profit)

proof by

- $\lim_{p \to 1/2}$ E[# unfair bets], or
- solving linear recurrence:

$$e_n = p(1 + e_{n+1}) + q(1 + e_{n-1})$$

In-Class Problem

Problems 2,3,4