

6	9	13	7
12	10	8	
3	1	4	14
15	5	11	2

Partial Orders

6	9	13	7
12	10	8	
3	1	4	14
15	5	11	2

Types of Relations

Equivalence

- Reflexive, Transitive, *Symmetric*.

Partial Orders

- (Reflexive), Transitive, *Antisymmetric*.

6	9	13	7
12	10	8	
3	1	4	14
15	5	11	2

Antisymmetric

$$\forall a, b \quad a \neq b \wedge aRb \rightarrow \neg(bRa)$$

Maybe aRb or bRa , **never both**.

- Either places an **order** on a, b
- Or a and b are **unrelated** (*incomparable*)

6	9	13	7
12	10	8	
3	1	4	14
15	5	11	2

Partial Order (Poset (\mathbb{Z}, \leq))

$$a R b \text{ if } a \leq b, a, b \in \mathbb{Z}$$

- Reflexive **YES**
- Transitive **YES**
- Symmetric **NO**
- Antisymmetric **YES**

- in fact a Total Order

6	9	13	7
12	10	8	
3	1	4	14
15	5	11	2

Total Order

- A relation R on a set A is a total order if it is a partial order and **for any a, b in A , either aRb or bRa**

–For any two distinct integers, x and y , either $x < y$ or $y < x$

6	9	13	7
12	10	8	
3	1	4	14
15	5	11	2

Not a Total Order: Divisibility

$$a R b \text{ if } a \mid b, a, b \in \mathbb{Z}$$

- $3 \mid 9$, but not $9 \mid 3$
- But 5 and 9 are not related (**incomparable**)

6	9	13	7
12	10	8	
3	1	4	14
15	5	11	2

Symmetric vs Antisymmetric

Symmetric:

$$\forall a, b \quad aRb \rightarrow bRa$$

Antisymmetric:

$$\forall a, b \quad a \neq b \wedge aRb \rightarrow \neg(bRa)$$

6	9	13	7
12	10	8	
3	1	4	14
15	5	11	2

EXERCISE

- Question: Can R be **neither** symmetric nor antisymmetric?
- YES: $R ::= \{(1,2) (2,1) (1,4)\}$
- Question: Can R be **both** symmetric and antisymmetric?
- YES: $R ::= \{(1,1)\}$

6	9	13	7
12	10	8	
3	1	4	14
15	5	11	2

A Relation on Classes

$cRd ::=$ class c is listed as a prerequisite to class d in the 6-3 curriculum



6	9	13	7
12	10	8	
3	1	4	14
15	5	11	2

A Partial Order?

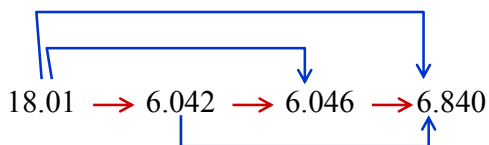
$cRd ::=$ class c is listed as a prerequisite to class d in the 6-3 curriculum

- Reflexive **No**
- Transitive **No**
- Antisymmetric **Yes**
(*certainly hope so!*)

6	9	13	7
12	10	8	
3	1	4	14
15	5	11	2

Creating a Partial Order

- Reflexive and Transitive closure of R



6	9	13	7
12	10	8	
3	1	4	14
15	5	11	2

A Relation on Classes

$cRd ::=$ class c is listed as a prerequisite to class d in the 6-3 curriculum



6	9	13	7
12	10	8	
3	1	4	14
15	11	5	2

Prerequisites

- 18.01 → 6.042
- 18.01 → 18.02
- 18.01 → 18.03
- 8.01 → 8.02
- 6.001 → 6.034
- 6.042 → 6.046
- 18.03, 8.02 → 6.002
- 6.001, 6.002 → 6.004
- 6.001, 6.002 → 6.003
- 6.004 → 6.033
- 6.033 → 6.857
- 6.046 → 6.840

6	9	13	7
12	10	8	
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15	11	5	2

Scheduling Problems

- How many terms will it take to graduate?
- How many classes does one need to take each term?
- Who's going to plan the whole thing?

- build a *Dependency Graph*

6	9	13	7
12	10	8	
3	1	4	14
15	11	5	2

Identify Minimal Elements

- **18.01** → 6.042
- 18.01 → 18.02
- 18.01 → 18.03
- **8.01** → 8.02
- **6.001** → 6.034
- 6.042 → 6.046
- 18.03, 8.02 → 6.002
- 6.001, 6.002 → 6.004
- 6.001, 6.002 → 6.003
- 6.004 → 6.033
- 6.033 → 6.857
- 6.046 → 6.840

6	9	13	7
12	10	8	
3	1	4	14
15	11	5	2

Directed Acyclic Graph (DAG)

18.01

8.01

6.001

6	9	13	7
12	10	8	
3	1	4	14
15	11	5	2

Prerequisites

- **18.01** → **6.042**
- 18.01 → **18.02**
- 18.01 → **18.03**
- **8.01** → **8.02**
- **6.001** → **6.034**
- 6.042 → 6.046
- 18.03, 8.02 → 6.002
- 6.001, 6.002 → 6.004
- 6.001, 6.002 → 6.003
- 6.004 → 6.033
- 6.033 → 6.857
- 6.046 → 6.840

6	9	13	7
12	10	8	
3	1	4	14
15	11	5	2

Directed Acyclic Graph (DAG)



6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Prerequisites

- 18.01 → 6.042
- 18.01 → 18.02
- 18.01 → 18.03
- 8.01 → 8.02
- 6.001 → 6.034
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- 18.03, 8.02 → 6.002
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- 6.001, 6.002 → 6.003
- 6.004 → 6.033
- 6.033 → 6.857
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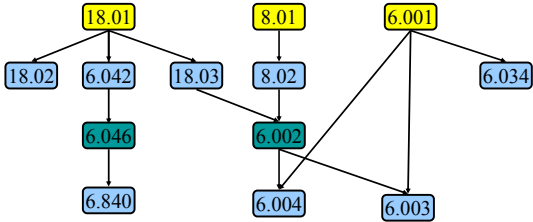
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12	10	5	
3	1	4	14
15	8	11	2

Directed Acyclic Graph (DAG)



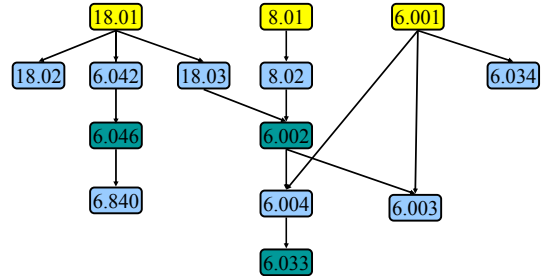
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12	10	5	
3	1	4	14
15	8	11	2

Directed Acyclic Graph (DAG)



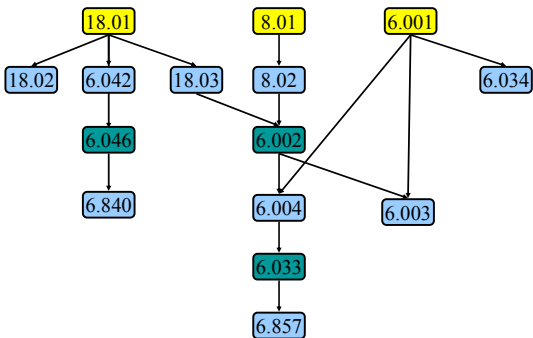
6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Directed Acyclic Graph (DAG)



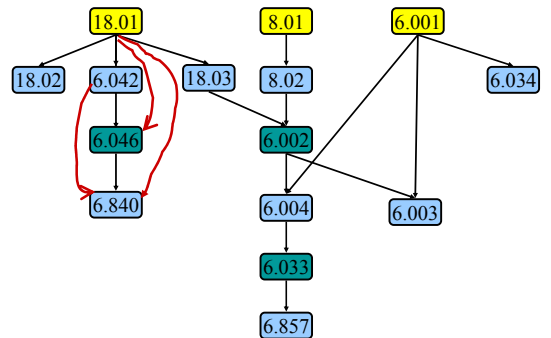
6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Directed Acyclic Graph (DAG)



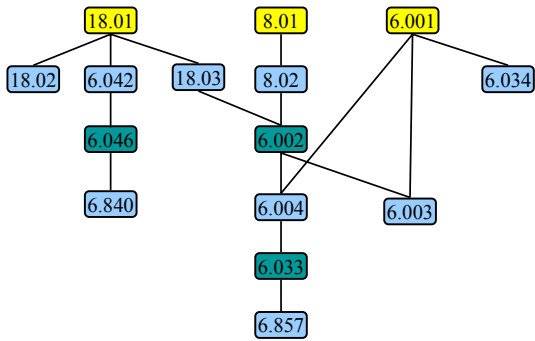
6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Partial Order (transitive edges)



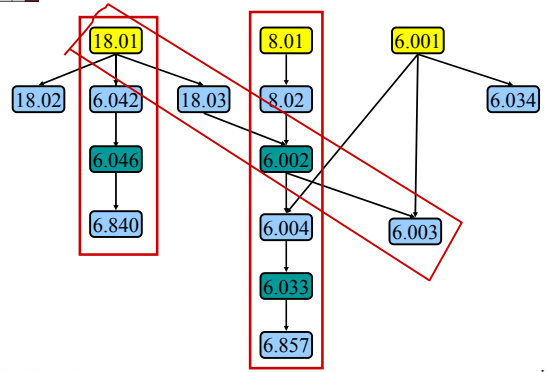
6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Hasse Diagram (no transitive edges)



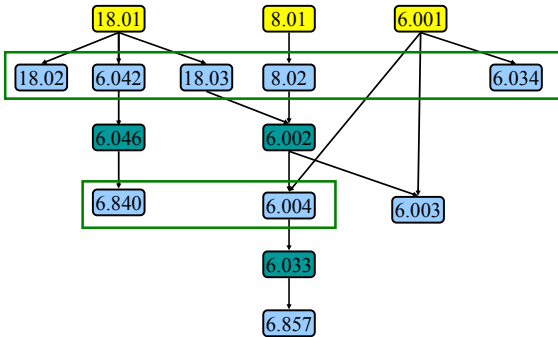
6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Chains



6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Anti Chains



6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

In-Class Problem 1

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Scheduling Problems

- How many terms will it take to graduate?
- How many classes does one need to take each term?
- Who's going to plan the whole thing?

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

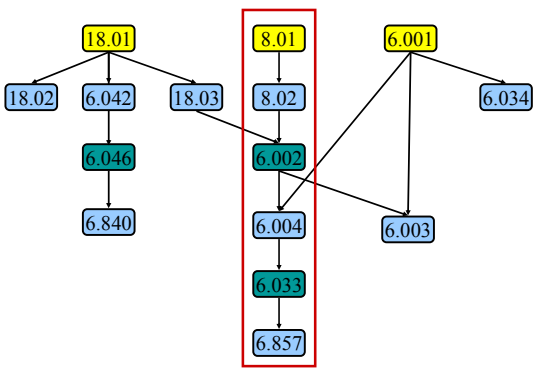
Parallel Task Scheduling

Theorem: If the longest chain has size t , then the elements can be partitioned into t antichains.

- 6 terms are **necessary** to complete the curriculum
- and **sufficient** (if you can take unlimited courses per year...)

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Necessary: Maximal Length Chain

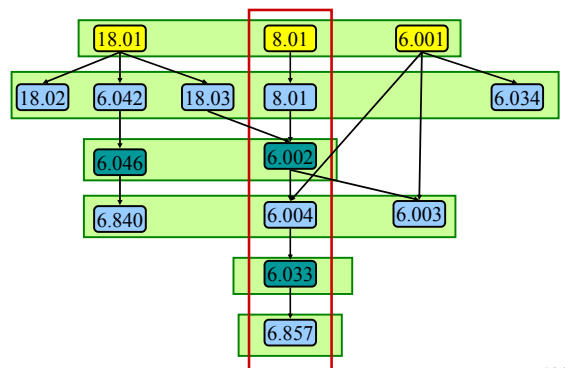


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L3-2.31

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Sufficient: Schedule

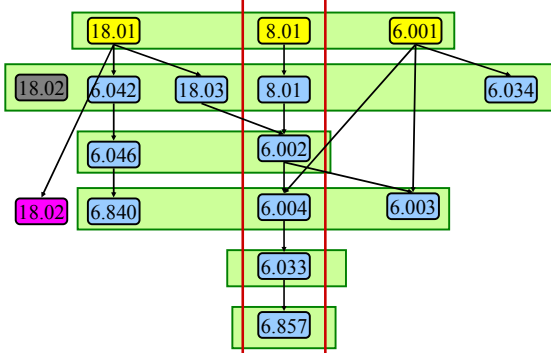


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L3-2.32

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Another Schedule



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L3-2.33

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Scheduling problems

- Real problem far more complex:
 - Limited per term, scheduling conflicts
- Other examples:
 - Stata Center construction
 - Program dependency graph

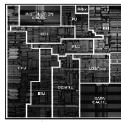


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L3-2.34

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

RS/6000 vs Alpha (IBM vs DEC)



- Superscalar: Hardware parallelism
- Alphas: Increase pipelining and clock speed
- Longest chain = minimum time to complete the program, *no matter how parallel your machine is!*

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L3-2.35

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Dilworth's Theorem

Theorem: Every poset of n elements has

- Either a **chain** of size at least t ,
- or an **antichain** of size at least $\left\lceil \frac{n}{t} \right\rceil$

for all $1 \leq t \leq n$.

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L3-2.36

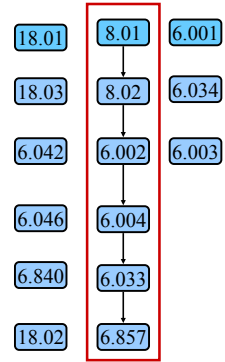
6	9	13	7
12	10	8	
3	1	4	14
15	5	11	2

Prereqs Graph

- Total **15** classes
- Length of maximum chain = **6**
- No chain of length 7
- → must be *at least one antichain with size at least 3*.
 – (at least one term where you have to take ≥ 3 classes)

6	9	13	7
12	10	8	
3	1	4	14
15	5	11	2

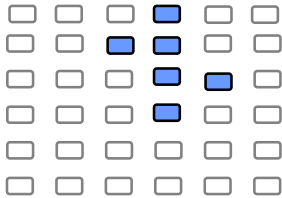
Classes per Term



6	9	13	7
12	10	8	
3	1	4	14
15	5	11	2

Dilworth's Theorem: Example

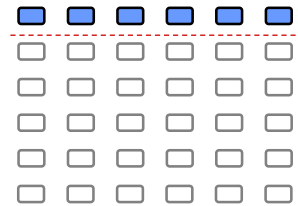
Consider a set S , where $|S| = n = 6$
 Column = chain, Row = anti-chain



6	9	13	7
12	10	8	
3	1	4	14
15	5	11	2

Dilworth's Theorem: Example

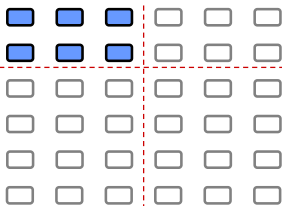
$|S| = n = 6$
 Longest chain $t = 1$



6	9	13	7
12	10	8	
3	1	4	14
15	5	11	2

Dilworth's Theorem: Example

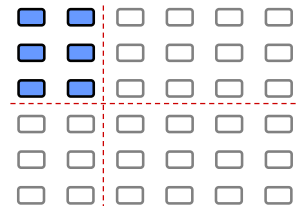
$|S| = n = 6$
 Longest chain $t = 2$



6	9	13	7
12	10	8	
3	1	4	14
15	5	11	2

Dilworth's Theorem: Example

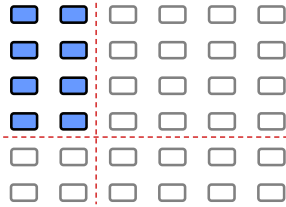
$|S| = n = 6$
 Longest chain $t = 3$



6	9	13	7
12	10	8	
3	1	4	14
15	5	11	2

Dilworth's Theorem: Example

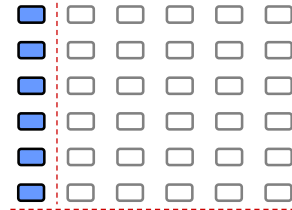
$|S| = n = 6$
Longest chain $t = 4$



6	9	13	7
12	10	8	
3	1	4	14
15	5	11	2

Dilworth's Theorem: Example

$|S| = n = 6$
Longest chain $t = 6$



6	9	13	7
12	10	8	
3	1	4	14
15	5	11	2

Dilworth's Theorem

Theorem: Every poset of n elements has

- Either a **chain** of size at least t ,
 - or an **antichain** of size at least $\left\lceil \frac{n}{t} \right\rceil$
- for all $1 \leq t \leq n$.

6	9	13	7
12	10	8	
3	1	4	14
15	5	11	2

In-Class Problems