Massachusetts Institute of Technology
6.042J/18.062J, Fall '02: Mathematics for Computer Science
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Solutions to In-Class Problems — Week 5, Fri

Problem 1. Consider the following game for two players. The players alternate moves. A move consists of a pair (x, y) of natural numbers, subject to the constraint that none of the previous moves may be $\preceq_c$ than the current move, where $\preceq_c$ is the coordinatewise partial order. An equivalent way to say this is that the current move must be $\prec_c$ or incomparable to all the previous moves. A player who moves to the origin $(0, 0)$ is the loser.

For example, the Player 1 might choose (5,6), after which Player 2 can move to any point (n, m) such that $n < 5$ or $m < 6$, for example, (4,12). Now the players might move successsively to (4,11), (29,5), (1,1), (0,54), (0,1). At this point it's Player 2's turn, and he can move to to (1,0). At this point it is Player 1's move, and the only available move is to the origin (0,0), so Player 1 loses this play of the game.

(a) Identify a winning strategy for the first player, and argue its correctness. Does your strategy guarantee any bound on the number of game moves?

Solution. The first player essentially has two winning strategies:

Strategy 1: The first player starts with $(1, 1)$. Then the second player can only pick $(0, n)$ or $(n, 0)$ for some n. The first player then responds with, $(n, 0)$ or $(0, n)$, respectively. By symmetry, the first player will always have a move whenever the second player had a move, except when the second player picks $(0, 0)$ and loses.

Strategy 2: The second winning strategy is slightly more tricky: The first player starts by picking $(0, 2)$,[1] and then:

- If the 2nd player picks $(0, n)$ for any $n \geq 1$, 1st player responds with $(1, n - 1)$.
- If the 2nd player picks $(1, n)$ for any $n \geq 0$, 1st player responds with $(0, n + 1)$.

The above procedure should be repeated for all consecutive moves. Notice that if Player 2 had a valid move, so does Player 1, unless Player 2 picked $(0, 0)$ and lost. ■

(b) Is there any strategy that guarantees a bound on the number of game moves?

[1]Of course, picking $(2, 0)$ would be equivalent, and the following argument will hold for it as well, with x and y coordinates replaced

Solution. No, because neither of the winning strategies described in the solution to part (a) guarantees a bound, and there are no other winning strategies.

To see this, suppose Player 1 starts with any legal move other than the ones above, Player 2 will be able to adopt one of the two winning strategies for himself and win for sure:

- If Player 1 picks some (m, n) where $m, n \geq 1$ and $(m, n) \neq (1, 1)$, then player 2 can pick $(1, 1)$ next, and then use the first strategy to win against Player 1.
- If Player 1 picks some $(0, n)$ or $(n, 0)$ where $n \geq 3$, then Player 2 can respectively pick $(0, 2)$ or $(2, 0)$ next, and then use the second strategy to win against Player 1.
- If Player 1 picks $(1, 0)$ or $(0, 1)$ then Player 2 picks the other one and wins.

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(c) Prove that the game must always terminate, even if the players conspire to choose strategies that try to keep the game going indefinitely.

Solution. This is a tricky problem without some hints.

Hint: At any point in the game, let x_m be the minimum of the values of the x coordinates of any of the previous moves, and likewise, y_m be the minimum of the values of the y coordinates of any of the previous moves. Then $x_m + y_m$ is a *weakly* decreasing natural-number valued variable.

Proof. That's because min's cannot increase as more moves are made, so neither can their sum. □

Hint: Suppose a is the least number such that a move (x_m, a) has been made, and likewise b is least such that (b, y_m) has been made. Then the only possible moves that do not decrease $x_m + y_m$ must occur in the rectangle with corners at (x_m, a) and (b, y_m).

Proof. Moves North of the rectangle are not allowed because they would be coordinatewise larger than (x_m, a), moves East of the rectangle are not allowed because they would be coordinatewise larger than (b, y_m), and moves Northeast would be bigger than both. Moves South of the rectangle decrease the minimum value of y, moves West decrease the minimum x, and moves Southwest decrease both. That leaves only moves within the rectangle as possibly allowed moves that do not decrease $x_m + y_m$. □

Hint: Define the size of a game position to be $(x_m + y_m, k)$ where k is the number of possible moves left in the rectangle.

Proof. Now size is a decreasing variable under the lexicographic ordering on $\mathbb{N}^2$. Since this lexicographic order is well-founded, the size of the states reached in any game must have a minimal value. Such a value must occur at a state in which the game is terminated. □

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NOTE October 4, 2002, 6PM: We didn't get to either of the following two problems.

Problem 2. In Week 5 Notes we considered win-lose games, ignoring games which can end in a draw. Such games have finite-path game trees in which the leaves are labelled with *win*, *lose*, or *draw*, indicating the outcome for the first player.

While a winning stategy for a player ensures the player will win no matter what moves the other player makes, there is now the possibility of a *non-losing* stategy which ensures that the player will *win or draw* no matter what moves the other player makes. (A winning strategy counts as a kind of a non-losing strategy.)

For example, in tic-tac-toe there is no winning strategy for either player, but *both* players have non-losing strategies. For chess, no one knows if white or black has a winning strategy. However, on general principles, we can be sure that at least one player has a non-losing chess strategy.

Explain why for any two-person win-lose-draw game with a finite path game tree, either one player has a winning strategy or *both* players have non-losing strategies.

Solution. We could prove this the same way we proved the Fundamental Theorem for win-lose games, using the well-foundedness of finite-path trees under the "strict-subtree" partial order, but it's more elegant to deduce this as a Corollary of the Fundamental Theorem.

Call the players C and D, and suppose neither one has a winning strategy for the game, T. We want to show that then *both* players have non-losing strategies for T.

The proof is based on the simple observation that in a game with no draws, a non-losing strategy must be a winning strategy.

To prove that C has a non-losing strategy for the game, T, let's consider another game, T_C, identical to T except that draws are considered wins for player C. Since there are no draws in T_C, we conclude that either C or D has a winning strategy for T_C. But a winning strategy for D in the T_C game would also be a winning strategy for D in the original game, T, because wins for D are the same in both games. But we assumed that D has no winning strategy for T, so it doesn't have one for T_C.

That means that C must have the winning strategy for T_C. But a winning strategy for C in T_C would also be a non-losing strategy for C in the original game, T, because wins for C in T_C are always wins for C or draws in T. So we conclude that C has a non-losing strategy for T.

Notice that we haven't assumed that C is the first or second player, so the preceding argument applies equally well to show that D also has a non-losing strategy for T, as required. ■

Problem 3. Prove part 2. of Lemma 3.4 below.

Solution. From Week 5 Notes, Lemma 8.4:

Proof. Suppose $\emptyset \neq S \subseteq P_1 \times P_2$. Then the set

$$S_1 ::= \{p_1 \in P_1 \mid (p_1, p_2) \in S \text{ for some } p_2 \in P_2\}$$

is a nonempty subset of P_1, and so has a $\preceq_1$-minimal element, m_1. This means the set

$$S_{12} ::= \{p_2 \in P_2 \mid (m_1, p_2) \in S\}$$

is a nonempty subset of P_2 and so has a $\preceq_2$-minimal element, m_2. We claim that (m_1, m_2) is a minimal element of S under *both* the coordinatewise and the lexicographic partial orders on $P_1 \times P_2$.

To check this, we consider any element $(n_1, n_2) \in S$ such that $(n_1, n_2) \preceq (m_1, m_2)$ and prove that $(n_1, n_2) = (m_1, m_2)$, where $\preceq$ may be either coordinatewise or lexicographic order.

Now we know that

$$n_1 \preceq_1 m_1$$

by definition of $\preceq$. Also $n_1 \in S_1$, so

$$m_1 \preceq_1 n_1$$

by definition of m_1. Hence,

$$n_1 = m_1.$$

This means that $n_2 \in S_{12}$, so

$$m_2 \preceq_2 n_2$$

by definition of m_2. But since $n_1 = m_1$, we have by definition of $\preceq$, that

$$n_2 \preceq_2 m_2,$$

proving that

$$n_2 = m_2,$$

as claimed. □

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A From Week 5 Notes

Definition 3.1. A poset $(P, \preceq)$ is *well-founded* iff every nonempty subset $S \subseteq P$ has a *minimal element*.

Lemma 3.2. *A poset is well-founded iff it has no infinite decreasing chain.*

Definition 3.3. Let $(P_1, \preceq_1)$ and $(P_2, \preceq_2)$ be partial orders. The *lexicographic partial order*, $\preceq_{\text{lex}}$, on $P_1 \times P_2$ is defined by the condition that

$$(p_1, p_2) \preceq_{\text{lex}} (q_1, q_2) \text{ iff } [p_1 \prec_1 q_1 \text{ or } (p_1 = q_1 \wedge p_2 \preceq_2 q_2)]. \tag{1}$$

The *coordinatewise partial order*, $\preceq_{\text{c}}$, on $P_1 \times P_2$ is defined by the condition that

$$(p_1, p_2) \preceq_{\text{c}} (q_1, q_2) \text{ iff } [p_1 \preceq_1 q_1 \wedge p_2 \preceq_2 q_2]. \tag{2}$$

Lemma 3.4. *Suppose $(P_1, \preceq_1)$ and $(P_2, \preceq_2)$ are posets. Then*

1. *so are $(P_1 \times P_2, \preceq_{lex})$ and $(P_1 \times P_2, \preceq_c)$. Moreover,*
2. *if $(P_1, \preceq_1)$ and $(P_2, \preceq_2)$ are both well-founded, then so are $(P_1 \times P_2, \preceq_{lex})$ and $(P_1 \times P_2, \preceq_c)$.*
3. *if $(P_1, \preceq_1)$ and $(P_2, \preceq_2)$ are both totally ordered, then so is $(P_1 \times P_2, \preceq_{lex})$.*