Random Variables and Expectation

1 Random Variables

When we perform an experiment, we expect the results to be observable—did the player hit a home run or not?—or measurable—how far did the ball travel? how fast was the pitch? To describe the behavior of such probabilistic experiments with measurable outcomes, we use *random variables*.

For example, consider the experiment of tossing three independent, unbiased coins. We can define C to be the number of heads which appear, and M to be 1 iff all three coins match and 0 otherwise. Any outcome of the coin flips uniquely determines C and M. C can take the values 0,1,2, and 3, and M the values 0 an 1.

We use the notation $[C = 2]$ for the event that there are two heads. Similarly, $[C \geq 2]$ is the event that there are at least two heads, and $[C \in \{1, 3\}]$ is the event that there are an odd number of heads.

Now consider the event that the product of C and M is positive; we write this one as $[C \cdot M > 0]$. Since neither C nor M take negative values, $C \cdot M >$ iff both $C > 0$ and $M > 0$—in other words, there is a head, and all three dice match. So saying $C \cdot M > 0$ is just an obscure way of saying that all three coin flips come up heads. That is, the event $[C \cdot M > 0]$ consists of the single outcome HHH.

When the meaning is clear, we often omit the square brackets denoting events. For example, we say "the event $C = 0$" instead of "the event $[C = 0]$," or $\Pr\{C = 0\}$ instead of $\Pr\{[C = 0]\}$.

Saying that each outcome uniquely determines C and M means that we can think of C and M as functions from outcomes to their values. The natural sample space, $\mathcal{S}$, for this experiment consists of eight outcomes: HHH, HHT, HTH, *etc*. For example, $C(\texttt{HHH}) = 3$, $C(\texttt{HTH}) = 2$, $C(\texttt{TTT}) = 0$. Similarly, $M(\texttt{HHH}) = 1$, $M(\texttt{HTH}) = 0$, $M(\texttt{TTT}) = 1$.

We can formalize the idea of a random variable in general as follows.

Definition 1.1. A *random variable* over a given sample space is a function that maps every outcome to a real number.

Notice that calling a random variable a "variable" a misnomer: it is actually a function.

We will use the random variables C and M as continuing examples. Keep in mind that C ***c***ounts heads and M indicates that all coins ***m***atch.