

## Problem Set 4

**Reading:** [Week 4 Notes](#), Supplementary (optional) Reading: Rosen §7.1-7.4, 7.8, and 8.1

Note: This problem set contains a number of questions that will require you to write proofs. The goal is not only to have correct proofs, but also to make sure they are clear, orderly, and well-presented.

**Problem 1.** The EECS department has six committees that meet once a month. How many different meeting times must be used to ensure that no one is scheduled to be at two meetings at the same time? The committees are  $C_1 = \{\text{Abelson, Meyer, Karger}\}$ ,  $C_2 = \{\text{Meyer, Leiserson, Rivest}\}$ ,  $C_3 = \{\text{Abelson, Rivest, Karger}\}$ ,  $C_4 = \{\text{Leiserson, Rivest, Karger}\}$ ,  $C_5 = \{\text{Abelson, Meyer}\}$ ,  $C_6 = \{\text{Meyer, Rivest, Karger}\}$ . Explain how you arrived at the answer.

**Problem 2.** Let  $G$  be an undirected graph. Prove that there is a path from every vertex of odd degree to some other vertex of odd degree in  $G$ .

**Problem 3.** A 2-colorable graph is called *bipartite* (two pieces).

- (a) Prove by induction that every tree is bipartite.
- (b) Prove that a graph is bipartite *if and only if* every simple cycle is of even length.<sup>1</sup>
- (c) Explain why (a) follows immediately from (b).

**Problem 4.** The *complement* of an undirected graph  $G = (V, E)$  is the graph  $G^c = (V, E^c)$ . That is,  $G^c$  contains exactly those edges that are not in  $G$ .

(a) Let the graph  $H = (\{A, B, C, D, E, F\}, \{\{A, F\}, \{B, D\}, \{B, E\}, \{C, D\}, \{C, E\}, \{D, E\}\})$ . Draw the graph  $H$  and circle the connected components.

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<sup>1</sup>Reminder: for *iff* you need to prove both directions of the implication.

*Hint:* Can  $v$  have both an even and an odd length path to  $w$ ?

- (b) Let the graph  $J = (\{A, B, C, D\}, \{\{A, B\}, \{C, D\}\})$ . Draw the complement of  $J$ .
- (c) Prove the following theorem:

**Theorem 4.1.** *If an undirected graph  $G$  is not connected then  $G^c$  is connected.*

- (d) However the converse of this theorem is not true: if  $G^c$  is connected, then  $G$  is not necessarily disconnected. Give an example of a connected graph whose complement is also connected.

**Problem 5. The Touring Knight** A knight is a chess piece that can move either two spaces horizontally and one space vertically or one space horizontally and two spaces vertically. That is, a knight on square  $(x, y)$  can move to any of the eight squares  $(x \pm 2, y \pm 1), (x \pm 1, y \pm 2)$ , if these squares exist on the chessboard (which is normally  $8 \times 8$ ).

A knight's tour is a sequence of legal moves by a knight starting at some square and visiting each square exactly once. A knight's tour is called reentrant if there is a legal move that takes the knight from the last square of the tour back to where the tour began. We can model knight's tours using the graph that has a vertex for each square on the board, with an edge connecting two vertices if a knight can legally move between the squares represented by these vertices.

- (a) Show that finding a reentrant knight's tour on an  $m \times n$  chessboard is equivalent to finding a Hamilton circuit on the corresponding graph.
- (b) Show that the graph representing the legal moves of a knight on a  $m \times n$  chessboard, wherever  $m$  and  $n$  are positive integers, is *bipartite*.
- (c) Deduce that there is no reentrant knight's tour on an  $m \times n$  chessboard when  $m$  and  $n$  are both odd.

