Appendix

Contents

1 Induction

Axiom (Induction).

$$\frac{P(0), \quad \forall m \in \mathbb{N}\; P(m) \longrightarrow P(m+1)}{\forall n \in \mathbb{N}\; P(n).}$$

Axiom (Strong Induction).

$$\frac{P(0), \quad \forall n \in \mathbb{N}\, \forall m \leq n\; P(m) \longrightarrow P(m+1)}{\forall n \in \mathbb{N}\; P(n)}$$

Axiom (Least Number Principle). Every nonempty subset, $S \subseteq \mathbb{N}$, has a smallest element.

2 Relations

2.1 Equivalence

A binary relation, R, on a set A is

- *reflexive* if for every $a \in A$, aRa.
- *symmetric* if for every $a, b \in A$, aRb implies bRa.
- *transitive* if for every $a, b, c \in A$, aRb and bRc implies aRc.

R is an *equivalence relation* iff it is reflexive, symmetric and transitive.

2.2 Partial Order

A binary relation, R, on a set A is

- *areflexive* if for every $a \in A$, $\neg(aRa)$.
- *asymmetric* if for every $a, b \in A$, aRb implies $\neg(bRa)$.
- *anti-symmetric* if for every $a, b \in A$ aRb and bRa implies $a = b$.
- a *partial order* if it is reflexive, transitive and anti-symmetric.
- a *strict partial order* if it is areflexive and transitive.
- a *total order* if it is a partial order and for every $a \neq b \in A$ either aRb or bRa.

If R is a partial order, then the set A is called a *partially ordered set*. The set, A, together with the partial order, R, is called a *poset*, (A, R).

In this case,

- a is a *minimal element* of A if $\nexists b \in A[bRa \wedge b \neq a]$
- a is a *maximal element* of A if $\nexists b \in A[aRb \wedge b \neq a]$
- a subset of A is a *chain* iff it is totally ordered by R.
- elements $a_1, a_2 \in A$ are *incomparable* iff neither $a_1 R a_2$ nor $a_2 R a_1$ holds.
- a subset of A is an *anti-chain* iff its elements are pairwise incomparable.

A *topological sort* of a finite poset (A, R) is a total ordering of all the elements of A, $a_1, a_2, \cdots, a_n$ in such a way that for all $i < j$, either $a_i R a_j$ or a_i and a_j are incomparable.

Theorem. *Given any finite poset (A, R) for which the longest chain has length t, A can be partitioned into t antichains.*

Theorem (Dilworth). *For all t, every poset with n elements must have either a chain of size greater than t or an antichain of size at least n/t.*

2.3 Operations

If R is a relation on $A \times B$, then the *inverse of* R is the relation on $B \times A$ given by

$$R^{-1} = \{(b, a) \mid (a, b) \in R\}.$$

The *composition* of relations $R_1 \subseteq A \times B$ and $R_2 \subseteq B \times C$ is the relation

$$R_2 \circ R_1 = \{(a, c) \mid (\exists b)((a, b) \in R_1) \wedge ((b, c) \in R_2)\}.$$

A *path* in a relation R is a sequence $a_0, \ldots, a_k$ with $k \geq 0$ such that $(a_i, a_{i+1}) \in R$ for every $i < k$. We call k the *length* of the path.

Let R be a relation on the set A and $n \in \mathbb{N}$. Then

$$R^n = \begin{cases} R & \text{if } n = 1 \\ R^{n-1} \circ R & \text{if } n > 1 \end{cases}$$

Lemma. $R^n = \{(a, b) \mid$ *there is a length n path from a to b in R*$\}$.

2.4 Closure

The *closure* of relation R with respect to property P is the relation S that

1. contains R,
2. has property P, and
3. is contained in *any* relation satisfying (1) and (2). That is, S is the "smallest"relation satisfying (1) and (2).

Lemma. *Let R be a relation on the set A.*

- *The reflexive closure of R is $S = R \cup \{(a, a), \forall a \in A\}$.*
- *The symmetric closure of R is $S = R \cup R^{-1}$.*
- *The transitive closure of a relation R is the set*

$$S = \{(a, b) \in A \times A \textit{ given there is a path from } a \textit{ to } b \textit{ in } R\}.$$

The *reflexive and transitive closure* (also called the *connectivity relation*) of R is

$$R^* = \{(a, b) \in A \times A \mid \text{ there exists a path from } a \text{ to } b \text{ in } R\}.$$

or, equivalently, $R^* = \bigcup_{n=1}^{\infty} R^n$.

Lemma. *If R is a reflexive relation on A then $R^* = R^{|A|}$.*

3 Graphs

A *simple graph* is a pair of sets (V, E) called *vertices* and *edges*, respectively. An edge $\{a, b\} \in E$ is a set where $a, b \in V$ and $a \neq b$.

A *directed graph* or *digraph* is a pair of sets of (V, E) where $E \subseteq V \times V$. In other words, an edge $(a, b) \in E$ is an *ordered pair* of vertices.

Theorem. *The sum of the degrees of the vertices in a simple graph equals twice the number of edges.*

Theorem (Handshake). *In every graph, there are an even number of vertices of odd degree.*

An *Euler tour* of an undirected graph G is a circuit that traverses every edge of G exactly once.

Theorem. *If undirected graph G has Euler tour, then G is connected and every vertex has even degree.*

A *Hamiltonian path* of an undirected graph G is simple path that traverses every vertex exactly once.

3.1 Common graphs

The *empty graph* or *anticlique* on n vertices, is the graph $A_n = (\{v_1, v_2, \ldots, v_n\}, \emptyset)$.

The *line graph* on n vertices is the graph $L_n = (\{v_1, v_2, \ldots, v_n\}, \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}\})$.

The *cycle* on n vertices C_n is the L_n plus the edge $\{v_n, v_1\}$.

The *wheel* on $n \geq 4$ vertices W_n is the graph consisting of C_{n-1} plus an additional vertex connected to every other vertex.

The *complete graph* or *clique* on n vertices, K_n has an edge between every pair of vertices.

3.2 Graph coloring

The minimum number of colors needed to color the vertices of a graph G so that no two adjacent vertices are the same is called the *chromatic number* and is written $\chi(G)$.

Lemma. *A graph with maximum degree d_{max} can be colored with d_{max} + 1 colors.*

3.3 Trees

A *tree* is any simple graph $G = (V, E)$, such that any of these (equivalent) conditions hold:

- G is connected and $|E| = |V| - 1$.
- G is connected, but removing any edge from G leaves a disconnected graph.
- G is connected and acyclic.
- There is a unique simple path between any two distinct vertices of G.

A rooted tree is called *finite-path* iff it has no infinite paths away from the root.

4 State Machines

A *state machine* has three parts:

1. a nonempty set, Q, whose elements are called *states*,
2. a nonempty subset $Q_0 \subseteq Q$, called the set of *start states*,
3. a binary relation, δ, on Q, called the *transition relation*.

An *invariant* for a state machine is a predicate, P, on states, such that whenever $P(q)$ is true of a state, q, and $q \to r$ for some state, r, then $P(r)$ holds.

Theorem (Invariant Theorem). *Let P be an invariant predicate for a state machine. If P holds for all start states, then P holds for all reachable states.*

A derived variable $f : Q \to \mathbb{R}$ is *strictly decreasing* iff

$$q \to q' \text{ implies } f(q') < f(q).$$

Theorem. *If $f : Q \to \mathbb{N}$ is a strictly decreasing derived variable of a state machine, then the length of any execution starting at a start state q is at most $f(q)$.*

4.1 Well-founded Partial Orders

If $(P_1, \preceq_1)$ and $(P_2, \preceq_2)$ are posets, then the *lexicographic partial order*, $\preceq_{\text{lex}}$, on $P_1 \times P_2$ is defined by the condition that

$$(p_1, p_2) \preceq_{\text{lex}} (q_1, q_2) \ ::= \quad p_1 \prec_1 q_1 \text{ or } (p_1 = q_1 \wedge p_2 \preceq_2 q_2).$$

The *coordinatewise partial order*, $\preceq_{\text{c}}$, on $P_1 \times P_2$ is defined by the condition that

$$(p_1, p_2) \preceq_{\text{c}} (q_1, q_2) \ ::= \quad (p_1 \preceq_1 q_1 \wedge p_2 \preceq_2 q_2).$$

A poset $(P, \preceq)$ is *well-founded* iff every nonempty subset $S \subseteq P$ has a *minimal element*.

Lemma. *Suppose $(P_1, \preceq_1)$ and $(P_2, \preceq_2)$ are posets. Then*

1. *so are $(P_1 \times P_2, \preceq_{lex})$ and $(P_1 \times P_2, \preceq_c)$. Moreover,*
2. *if $(P_1, \preceq_1)$ and $(P_2, \preceq_2)$ are both well-founded, then so are $(P_1 \times P_2, \preceq_{lex})$ and $(P_1 \times P_2, \preceq_c)$.*
3. *if $(P_1, \preceq_1)$ and $(P_2, \preceq_2)$ are both totally ordered, then so is $(P_1 \times P_2, \preceq_{lex})$.*

Lemma. *A poset is well-founded iff it has no infinite decreasing chain.*

Theorem. ***Fundamental Theorem for two-person games of perfect information**: For games in which every play is finite and ends in win or lose, there is a winning strategy for one of the players.*