

6	9	13	7
12	10	5	
3	1	4	11
15	8	14	2

Great Expectations.III

6	9	13	7
12	10	5	
3	1	4	11
15	8	14	2

Sum of 3 Dice



- Roll 3 fair dice
- What is the expected sum?

$$R = R_1 + R_2 + R_3$$

$$\begin{aligned} E[R] &= E[R_1] + E[R_2] + E[R_3] \\ &= 3.5 \cdot 3 \end{aligned}$$

6	9	13	7
12	10	5	
3	1	4	11
15	8	14	2

Sum of 2 or 3 Dice



- Flip a fair coin
 - If **heads**, roll 3 fair dice
 - If **tails**, roll 2 dice
- What is the expected sum?

$$R = R_1 + R_2 + R_3 \text{ or } R_1 + R_2$$

6	9	13	7
12	10	5	
3	1	4	11
15	8	14	2

Conditional Expectation

$$E[R | A] = \sum_r r \cdot \Pr\{R = r | A\}$$

Law of Total Expectation

$$E[R] = \sum_i E[R | A_i] \cdot \Pr\{A_i\}$$

6	9	13	7
12	10	5	
3	1	4	11
15	8	14	2

Sum of 2 or 3 Dice



If **heads**, roll 3 fair dice,
if **tails**, roll 2 dice

$$E[R] = E[R | \text{heads}] \cdot \Pr\{\text{heads}\} + E[R | \text{tails}] \cdot \Pr\{\text{tails}\}$$

$$\begin{aligned} E[R] &= (3.5 \times 3) \cdot \frac{1}{2} \\ &\quad + (3.5 \times 2) \cdot \frac{1}{2} \\ &= 3.5 \times 2.5 \end{aligned}$$

6	9	13	7
12	10	5	
3	1	4	11
15	8	14	2

Roll a Dice,



- If = 1, then roll 1 dice
 - If = 2, then roll and sum 2 dice
 - If = 3, then roll and sum 3 dice
 -
 - If = 6, then roll and sum 6 dice
- What's the expected sum?



Wald's Theorem

Let $R = R_1 + R_2 + R_3 + R_4 + \dots + R_Q$
where Q is also a random variable

Then $E[R] = E[R_i] \times E[Q]$
(many restrictions apply)



Sum of 2 or 3 Dice



If **heads**, roll **3** fair dice,
if **tails**, roll **2** dice

$E[R_i]$ = expected value of a dice = 3.5
 $E[Q]$ = expected number of rolls = 2.5
By Wald's Theorem

$$E[R] = 3.5 \times 2.5$$

Same answer!



Useful Way for Analyzing Processes

Leader Election

Expected Time
= Time taken/round . $E[\#Rounds]$
(FIXED) $1/p_{\text{success}}$



Wald's Theorem

Different process:

Time taken/round is not fixed

Expected Time
= $E[\text{Time taken/round}] \cdot E[\#Rounds]$



Example:

- Flip a coin until get **3 heads** in a row
- Possible run: **HTTHTHTHTHHH**
- How many coin flips on average?



Example:

- Possible run: **HTTHTHTHTHHH**

How many rounds did I run?
 $E[Q]=?$
 $= 1/p_{\text{success}}$

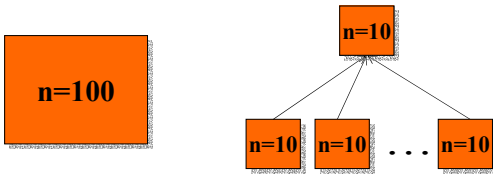
1: **HT** ← How many coin flips before I get a T?
2: T
3: HHT
4: HT
5: HHH

$E[R_i]=?$



A Case for Modular Design

Let T_n be the time to assemble n parts



$$E[T_{100}]$$

$$E[T_{100}] = 10 \cdot E[T_{10}]$$



Herbert Simon

The Architecture of Complexity, 1962



In Class Problem 1:

Find the Bug!!!



Fun with Infinite Series

Consider the Series:

- $1 - 1 + 1 - 1 + 1 - 1 + \dots$
- $(1 - 1) + (1 - 1) + (1 - 1) + \dots = 0$

But I could rearrange the grouping,

- $1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + \dots = 1 !!!$
- $= 2$
- $= 100$



Infinite Linearity of Expectations

$$E[C_1 + C_2 + C_3 + \dots]$$

$$= E[C_1] + E[C_2] + E[C_3] + \dots$$

But!!! It might converge to different values based on rearranging the terms!

(must check for absolute convergence) $\sum_{i=1}^{\infty} E[C_i]$



The Bet-Doubling Strategy

At any round r

- Place a bet = $\$10 \cdot 2^{r-1}$
- Total bet so far:
 - $= \$10 (2^0 + 2^1 + 2^2 + \dots + 2^{r-1})$
 - $= \$10 (2^r - 1)$
- If you win, receive: $\$10 \cdot 2^r$
- Net Win = $\$10$





What is the Expected Win

- **Correct Argument**
 - Expected number of rounds = $1/p = 2$
 - But probability of never getting red = 0
 - Win \$10 with certainty
- **Incorrect Argument**
 - Expected win in one round $E[C_i] = 0$
 - Therefore expected win $E\left[\sum_{r=1}^{\infty} C_i\right] = 0$



Problem: What's your worth?

- Let M = amount of money you had to put up
- At round r , $M = \$10 (2^r - 1)$
- $\Pr\{\text{win round on } r\} = (1-p)^{r-1}p = 1/2^r$

$$E[M] = \sum_{r=1}^{\infty} 10 \cdot (2^r - 1) \cdot \frac{1}{2^r} \approx \sum_{r=1}^{\infty} 10 = \text{infinite}$$



In Class Problems



In Class Problem 1 (c)

$$E[\text{win}] = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots = \infty.$$

What if the Casino is

limited to a Billion dollars (2^{30})?

$$E[\text{win}] = \sum_{i=1}^{30} 2^i \frac{1}{2^{i-1}} + \frac{1}{2^{30}} \cdot 2^{30}$$

(< 30 tails) (30 tails)

$$= 15 + 1 = 16!!!!$$



Infinite Expectation

- Random Variable, R
- Probability Density, $f_R(r) ::= \Pr\{R=r\}$

$$\sum_{r \in \text{range}(R)} \Pr\{R=r\} = 1 \quad \leftarrow \text{MUST BE TRUE}$$

$$E[R] ::= \sum_{r \in \text{range}(R)} r \cdot \Pr\{R=r\} \quad \leftarrow \text{MIGHT BE INFINITE}$$



Example:

$$R = 2, 4, 8, 16, 32, \dots$$

where

$$\Pr\{R = 2^i\} = \frac{1}{2^i}$$

6	9	13	7
12	10	5	
3	1	4	11
15	8	14	2

$$\begin{aligned} & \sum_{i=1}^{\infty} \Pr\{R = 2^i\} \\ &= \sum_{i=1}^{\infty} \frac{1}{2^i} \\ &= \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i - 1 \\ &= 2 - 1 = 1 \end{aligned}$$

6	9	13	7
12	10	5	
3	1	4	11
15	8	14	2

$$\begin{aligned} & E[R] \\ &= \sum_{i=1}^{\infty} r \Pr\{R = r\} \\ &= \sum_{i=1}^{\infty} 2^i \frac{1}{2^i} \\ &= \sum_{i=1}^{\infty} 1 \leftarrow \textit{infinite} \end{aligned}$$