Massachusetts Institute of Technology
6.042J/18.062J, Fall '02: Mathematics for Computer Science
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In-Class Problems — Week 2, Wed

Problem 1. (a) Prove that a $2^n \times 2^n$ plaza can be tiled using L-shaped tiles leaving untiled *any* single 1×1 square desired. (Yes, the solution was in the reading, but do you remember it, and can you explain/write it?)

(b) If you finish part (a) fast, you can think about what happens if the area of the plaza is not a power of two. For example, can you characterize exactly which $m \times n$ rectangles can be L-tiled with a single empty square where? (We don't know a complete classication of the L-tilable rectangles.)

Problem 2. Pinpoint, and illustrate with a counterexample, *exactly* where the following proof goes awry.

False Claim. $\forall n \in \mathbb{N} \, \forall x \in \mathbb{R} \;\; x^n = 1.$

False proof. (by induction on n)

Induction Hypothesis:

$$P(n) ::= \;\; \forall k \leq n \, \forall x \in \mathbb{R} \;\; x^k = 1.$$

Base case ($n = 0$): $P(0)$ is $\forall x \in \mathbb{R} \; x^0 = 1$. This holds by definition of "zeroth power" of a real number.

Induction step: Assuming $P(n)$, we have $x^k = 1$ for all $k \leq n$. To prove $P(n+1)$ we must show that $x^k = 1$ for all $k \leq n+1$. Since the induction hypothesis already handles $k \leq n$, we need only show that $x^{n+1} = 1$. But applying the induction hypothesis with $k = n$ and $k = 1$, we have $x^n = 1$ and $x^1 = 1$. So,

$$x^{n+1} = x^n x^1 = 1 \times 1 = 1.$$

□

Problem 3. Prove by induction on the size of finite sets, A, B, that

$$|A \cup B| = |A| + |B| - |A \cap B| \qquad (1)$$