



Introduction to Random Variables



Guess the Bigger Number

Team 1:

- Write different integers between 0 and 7 on two pieces of paper
- Show to Team 2 face down

Team 2:

- Expose one number
- Either *stick*, or *switch* to other number
- Expose other number
- win if end with *larger* number



Guess the Bigger Number

Try it out!



Strategy for Team 2

Choose papers with equal probability.
If exposed number is “small” then *switch*; otherwise *stick*.

“small” means \leq threshold Z .

Z is *random integer*, $0 \leq Z \leq 6$.



Analysis of Team 2 Strategy

Case 1 ($low \leq Z < high$):

$$\Pr\{Z = low\} = 1/7, \text{ so}$$

$$\Pr\{\text{Case 1}\} \geq 1/7.$$

Team 2 wins in this case, so

$$\Pr\{\text{Team 2 wins} \mid \text{Case 1}\} = 1.$$



Analysis of Team 2 Strategy

Case 2 ($high \leq Z$):

Team 2 wins iff low card exposed.

$$\Pr\{\text{Team 2 wins} \mid \text{Case 2}\} = 1/2.$$



Analysis of Team 2 Strategy

Case 3 ($Z < low$):

Team 2 wins iff high card exposed.

$$\Pr\{\text{Team 2 wins} \mid \text{Case 3}\} = 1/2.$$



Analysis of Team 2 Strategy

At least 1/7 of time, sure win.

Rest of time, 50/50 win, so

$$\Pr\{\text{Team 2 wins}\} \geq \frac{1}{7} \cdot 1 + \frac{6}{7} \cdot \frac{1}{2} = \frac{4}{7} > \frac{1}{2}$$



Analysis of Team 2 Strategy

Does not matter what
Team 1 does!!



In-class Problem: Team 1 Strategy

How can Team 1 guarantee

$$\Pr\{\text{Team 2 wins}\} \leq 4/7$$

no matter what Team 2 does?

Prove it.



Random Variables

Examples of *random variables*:

- Number of larger card
- Number of smaller card
- Number of exposed card
- “threshold” number Z

Intuitive definition:

a number produced by
a “random process”



Intro to Random Variables

Example: Flip three fair coins.

$C ::=$ number of heads (count).

$$M ::= \begin{cases} 1 & \text{if all match,} \\ 0 & \text{otherwise.} \end{cases}$$



Intro to Random Variables

Specify events using values of variables.

- $[C=1]$ is the event “exactly 1 head”

$$\Pr\{[C = 1]\} = 3/8$$
- $\Pr\{[C \geq 1]\} = 7/8$
- $\Pr\{[C \cdot M > 0]\} = \Pr\{[M > 0 \text{ and } C > 0]\}$

$$= \Pr\{\text{all heads}\} = 1/8$$



Density & Distribution

- *probability density function* of R :

$$f_R(r) ::= \Pr\{R = r\}$$

- *(Cumulative) distribution function*:

$$F_R(r) ::= \Pr\{R \leq r\}$$



Uniform Distribution

R is *uniform* iff f_R is constant.

Examples: $R_1 ::=$ outcome of fair die.

$$\Pr\{R_1=1\} = \dots = \Pr\{R_1=6\} = 1/6$$

$R_2 ::=$ 4 digit lottery number

$$\Pr\{R_2 = 0000\} = \Pr\{R_2 = 0001\} = \dots =$$

$$\Pr\{R_2 = 9999\} = 1/10000$$



Uniform Distribution

“Threshold” variable Z :

$$f_Z(i) ::= \Pr\{Z = i\} = 1/7,$$

$$F_Z(i) ::= \Pr\{Z \leq i\} = (i+1)/7,$$

for $i = 0, 1, \dots, 6$.



What is a Random Variable?

Formally, an RV, R , is any function:

$$R: \underbrace{\mathcal{S}}_{\text{Sample Space}} \rightarrow \text{Reals.}$$



In-class Problems

Problems 1 & 2
 (Lecture ended here)

6	9	13	7
12	10	5	
3	1	4	11
15	8	14	2

Independent Variables

Random variables R, S are *independent* iff
 $[R = a]$ and $[S = b]$
 are independent *events* for all
 real a, b .

6	9	13	7
12	10	5	
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Independent Variables

Alternative version: R, S *independent* iff
 $\Pr\{R = a \mid S = b\} = \Pr\{R = a\}$.

Alternative version 2:

$$\Pr\{R = a \text{ and } S = b\} = \Pr\{R = a\} \cdot \Pr\{S = b\}.$$

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Indicator Variables

Event A has *indicator variable*

$$I_A ::= \begin{cases} 1 & \text{if } A \text{ occurs,} \\ 0 & \text{if } \overline{A} \text{ occurs.} \end{cases}$$

Sanity check:

I_A and I_B are independent iff
 A and B are independent.

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Independent Variables

Mutual Independence:

$$\Pr\{A_1=a_1 \text{ and } A_2=a_2 \dots \text{and } A_n=a_n\} = \Pr\{A_1=a_1\} \times \Pr\{A_2=a_2\} \times \dots \times \Pr\{A_n=a_n\}.$$

Pairwise Independence:

$$\Pr\{A_i=a_i \text{ and } A_j=a_j\} = \Pr\{A_i=a_i\} \times \Pr\{A_j=a_j\}.$$

all $i \neq j$.

6	9	13	7
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Independent Variables

Example:

$R_i ::=$ indicator for Head on coin i

$$P ::= R_1 + R_2 + R_3 \pmod{2}.$$

Not mutually independent:
 any 3 determine the 4th one.

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Independent Variables

Example:

$R_i ::=$ indicator for Head on coin i

$$P ::= R_1 + R_2 + R_3 \pmod{2}.$$

But **3-way independent**, e.g.,

$$\Pr\{P = 0 \text{ and } R_1 = 1 \text{ and } R_0 = 0\} = \Pr\{P = 0\} \times \Pr\{R_1 = 1\} \times \Pr\{R_0 = 0\}$$



Independent Variables

Pairwise Independence sufficient for major applications (in later lecture).

Good to know, since pairwise holds in important cases where mutual does not.



In-class Problems

Problem 3



Distributions

Bernoulli variable ::= 0-1 valued (indicator variable)

Example:

$M, R_i, (C \bmod 2)$ are Bernoulli, but **not** C .



Binomial Distribution

Binomial distribution is distribution of # heads in n mutually independent flips.

Example: C is binomial for 3 flips.

Coin may be **biased**. So 2 parameters

n ::= # flips, p ::= $\Pr\{\text{head}\}$.

C has binomial density $f_{3,1/2}$



Binomial Distribution

Alternative definition:

B is **binomial** (n,p) random variable iff

$$B = X_1 + X_2 + \dots + X_n$$

for $\{X_i\}$ **mutually independent**

Bernoulli with $\Pr\{X_i=1\} = p$.



Binomial Distribution

$\Pr\{k \text{ Heads in } n \text{ flips}\} =$

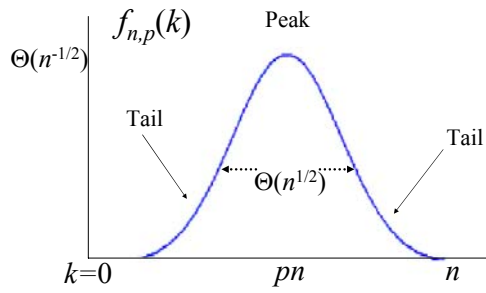
$$f_{n,p}(k) ::= \binom{n}{k} p^k q^{n-k}$$

binomial density function

$$q ::= 1 - p$$

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The Binomial PDF



6	9	13	7
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In-class Problems

Problem 4