Solutions to In-Class Problems — Week 14, Wed

Problem 1. A gambler has been studying the roulette wheel in a Las Vegas casino over a long period of time, and his data confirms that the wheel is biased to come up on a red number about 51% of the time. Since a bet on red pays even money (bet \$1 and either lose it, or win and get back \$2), he realizes the odds are in his favor.

So with high hopes and an initial stake of \$10,000, the gambler aims to turn his stake into \$1,000,000. He plans simply to continue making \$100 bets on red until either he is ahead the \$1,000,000, or he goes bankrupt. He considers bankruptcy very unlikely because the odds are in his favor, and the bets are small enough relative to his initial stake that he can withstand a long run of losses.

(a) Based on this description, describe a reasonable probability space to model this situation.

Solution. An easy way to describe an appropriate sample space is to keep track of the gambler's stake. So sample points will be all the finite sequences of nonnegative multiples of 100, that

- start at 10,000,
- end at the first occurrence of either 0 or 1,000,000,
- have successive integers in the sequence differ by exactly 100.

For example, the sample point describing two initial wins, followed by 102 losses and bankruptcy would be the length 105 sequence

$$(10000, 10100, 10200, 10100, 10000, 9900, 9800, \ldots, 200, 100, 0).$$

The sample point corresponding to his winning every bet until he reaches his goal would the length 9,900 sequence

$$(10000, 10100, 10200, 10300, 10400, \ldots, 999800, 999900, 1000000).$$

The probability of an outcome is defined to be $(0.51)^k(0.49)^j$ where k is the number of increases by 100, and j is the number of decreases by 100. Note that either $j - k = 100$ for sample points that lead to bankruptcy or $k - j = 9,900$ for those leading to \$1,000,000. ■

(b) Describe a positive constant, ϵ, such that, at any point during his play when the gambler has not gone bankrupt, there is a probability of at least ϵ that he will reach his goal of \$1,000,000 within the next 9,999 bets.

Solution. $\epsilon ::= (0.51)^{9999}$ will do, since even if the gambler's stake was down to \$100, there is still this much probability that he will win the next 9,999 bets in a row and reach his goal of \$1,000,000. ■

(c) Conclude that the probability that the gambler bets $n \geq 9,999$ or more times before the game ends is at most r^{n-9999} for some constant $r < 1$.

Solution. Let $r ::= (1-\epsilon)^{1/9999}$. The reason this works is that the probability of not reaching the goal of \$1,000,000 in the first 9999 bets is at most $1-\epsilon$, because there is at least an ϵ chance the Gambler wins by the 9,999th bet. Likewise, the probability of not reaching the goal of \$1,000,000 in the second 9999 bets—given that he didn't in the first 9999—is also $\leq 1-\epsilon$. So the probability of not reaching the goal in the first $2 \cdot 9999$ bets is $\leq (1-\epsilon)^2$. Repeating this argument for the first $\lfloor n/9999 \rfloor$ blocks of 9999 bets, we conclude that the probability of not reaching the goal of \$1,000,000 by the nth bet is $\leq$ the probability of not reaching it by the $\lfloor n/9999 \rfloor \cdot 9999 \leq n$th bet, which is $\leq (1-\epsilon)^{\lfloor n/9999 \rfloor} = r^{\lfloor n/9999 \rfloor \cdot 9999} \leq r^{n-9999}$. ■

NOTE: We didn't get to any of the following three problems in class.

Problem 2. The "high hopes" of the Gambler in Problem 1 overlook some difficulties with his strategy. Is he really very likely to win? How long will it take?

Solution. He has a 98% chance of winning: more than $1-(0.49/0.51)^{100}$ using the formula (7) bounding the probability of getting an intended profit in an unfair game. To apply the formula, we reverse the roles of T and n to turn the favorable game into an unfair one. This turns into a game with probability 0.49 of winning a bet, starting with a stake of 9900 betting units of \$100 and aiming to win 100 units. So the probability of a win in this game is a most $(0.49/0.51)^{100} \approx 0.018$; winning this reversed game corresponds to going bankrupt in the original game, so his probability of winning the real game is $\geq 1-0.18$.

It follows from the formula for expected number of bets that he can expect to make about a half million bets before winning, however. ■

Problem 3. Let G be the amount won by the gambler when a Gambler's Ruin game ends. Let Q be the number of bets till the game ends.

The derivation of the formula for the expected number of bets, $\text{E}[Q]$, in an unfair Gambler's Ruin game used the fact that

$$\text{E}[Q] \cdot \text{E}[\text{amount won win per bet}] = \text{E}[G]. \tag{1}$$

Prove this equation. *Hint:* Since the amount won per bet may be negative, Wald's Theorem does not immediately apply.

Solution. Directly from Notes 13-14:

Let G_i be the amount the gambler gains on the ith flip: $G_i = 1$ if the gambler wins the flip, $G_i = -1$ if the gambler loses the flip, and $G_i = 0$ if the game has ended before the ith flip. So the amount won by the gambler when the game ends is

$$G = \sum_{i=1}^{Q} G_i.$$

Now the random variable $G_i + 1$ is nonnegative, and $\mathrm{E}[G_i + 1 \mid Q \geq i] = \mathrm{E}[G_i \mid Q \geq i] + 1 = \mathrm{E}[\text{amount won win per bet}] + 1$, so by Wald's Theorem

$$\mathrm{E}\left[\sum_{i=1}^{Q}(G_i + 1)\right] = \mathrm{E}[\text{amount won win per bet}] + 1 \cdot \mathrm{E}[Q]. \tag{2}$$

But

$$\begin{aligned}
\mathrm{E}\left[\sum_{i=1}^{Q}(G_i + 1)\right] &= \mathrm{E}\left[\sum_{i=1}^{Q} G_i + \sum_{i=1}^{Q} 1\right] \\
&= \mathrm{E}\left[(\sum_{i=1}^{Q} G_i) + Q\right] \\
&= \mathrm{E}\left[\sum_{i=1}^{Q} G_i\right] + \mathrm{E}[Q] \\
&= \mathrm{E}[G] + \mathrm{E}[Q]. \qquad (3)
\end{aligned}$$

Now combining (2) and (3) confirms the truth of (1). ■

Problem 4. Prove that in an *unbounded* fair game, where the Gambler plays until he is broke no matter how much his stake increases in the meantime, the Gambler is *sure* to go broke, but the expected number of bets before he goes broke is infinite.

Solution. Directly from Notes 13-14:

Lemma 4.1. *If the gambler starts with one or more dollars and plays a fair game until he is broke, then he will go broke with probability 1.*

Proof. If the gambler has initial capital n and goes broke in a game without reaching a goal T, then he would also go broke if he were playing and ignored the goal. So the probability that he will lose if he keeps playing without stopping at any goal T must be at least as large as the probability that he loses when he has a goal $T > n$.

But we know that in a fair game, the probability that he loses is $1 - n/T$. This number can be made arbitrarily close to 1 by choosing a sufficiently large value of T. Hence, the probability of his losing while playing without any goal has a lower bound arbitrarily close to 1, which means it must in fact be 1. □

So even if the gambler starts with a million dollars and plays a perfectly fair game, he will eventually lose it all with probability 1.

Lemma 4.2. *If the gambler starts with one or more dollars and plays a fair game until he goes broke, then his expected number of plays is infinite.*

Proof. Consider the gambler's ruin game where the gambler starts with initial capital n, and let u_n be the expected number of bets for the *unbounded* game to end. Also, choose any $T \geq n$, and as above, let e_n be the expected number of bets for the game to end when the gambler's goal is T.

The unbounded game will have a larger expected number of bets compared to the bounded game because, in addition to the possibility that the gambler goes broke, in the bounded game there is also the possibility that the game will end when the gambler reaches his goal, T. That is,

$$u_n \geq e_n.$$

So by (8),

$$u_n \geq n(T - n).$$

But $n \geq 1$, and T can be any number greater than or equal to n, so this lower bound on u_n can be arbitrarily large. This implies that u_n must be infinite.

Now by Lemma 4.1, with probability 1, the unbounded game ends when the gambler goes broke. So the expected time for the unbounded game to *end* is the *same* as the expected time for the gambler to *go broke*. Therefore, the expected time to go broke is infinite. □

In particular, even if the gambler starts with just one dollar, his expected number of plays before going broke is infinite! Of course, this does not mean that it is likely he will play for long. For example, there is a 50% chance he will lose the very first bet and go broke right away.

Lemma 4.2 says that the gambler can "expect" to play forever, while Lemma 4.1 says that with probability 1 he will go broke. These Lemmas sound contradictory, but our analysis showed that they are not. ■

A Appendix

The *expectation* of random variable, R, is:

$$\mathrm{E}[R] ::= \sum_{r \in \mathrm{range}(R)} r \cdot \Pr\{R = r\}.$$

If R has codomain $\mathbb{N}$, then this definition can also be written as

$$\mathrm{E}[R] = \sum_{r \in \mathbb{N}} \Pr\{R > r\}$$

The *conditional expectation*, $\mathrm{E}[R \mid A]$, of a random variable, R, given event, A, is

$$\mathrm{E}[R \mid A] ::= \sum_{r} r \cdot \Pr\{R = r \mid A\}. \tag{4}$$

Theorem (Law of Total Expectation). *If the sample space is the disjoint union of events $A_1, A_2, \cdots$, then*

$$\mathrm{E}[R] = \sum_i \mathrm{E}[R \mid A_i] \Pr\{A_i\}.$$

Theorem (Wald). *Let $C_1, C_2, \ldots,$ be a sequence of nonnegative random variables, and let Q be a positive integer-valued random variable, all with finite expectations. Suppose that*

$$\mathrm{E}[C_i \mid Q \geq i] = \mu$$

for some $\mu \in \mathbb{R}$ and for all $i \geq 1$. Then

$$\mathrm{E}[C_1 + C_2 + \cdots + C_Q] = \mu \, \mathrm{E}[Q].$$

Theorem. *In the Gambler's Ruin game with probability p of winning each individual bet, with initial capital, n, and goal, T,*

$$\Pr\{\textit{the gambler is a winner in the fair game}\} = \frac{n}{T}, \tag{5}$$

$$\Pr\{\textit{the gambler is a winner a biased game}\} = \frac{(q/p)^n - 1}{(q/p)^T - 1}. \tag{6}$$

$$\Pr\{\textit{the gambler is a winner in an unfair game}\} \leq (p/q)^{T-n}. \tag{7}$$

Let Q be the number of bets till the game ends.

$$\mathrm{E}[Q \textit{ in an unfair game}] = \frac{\Pr\{\textit{gambler is a winner}\}\, T - n}{2p - 1}.$$

$$\mathrm{E}[Q \textit{ in a fair game}] = n(T - n). \tag{8}$$