

Solutions to In-Class Problems — Week 5, Wed

Problem 1. We apply the following operation to a simple graph G : pick two vertices $u \neq v$ such that either

1. there is an edge of G between u and v and there is also a path from u to v which does *not* include this edge; in this case, delete the edge $\{u, v\}$.
2. there is no path from u to v ; in this case, add the edge $\{u, v\}$.

Keep repeating these operations until it is no longer possible to find two vertices $u \neq v$ to which an operation applies.

Assume the vertices of G are the integers $1, 2, \dots, n$ for some $n \geq 2$. This procedure can be modelled as a state machine whose states are all possible simple graphs with vertices $1, 2, \dots, n$. G is the start state, and the final states are the graphs on which no operation is possible.

(a) Let G be the graph with vertices $\{1, 2, 3, 4\}$ and edges

$$\{\{1, 2\}, \{3, 4\}\}$$

What are the possible final states reachable from start state G ? Draw them.

Solution. It's not possible to delete any edge. The procedure can only add an edge connecting exactly one of vertices 1 or 2 to exactly one of vertices 3 or 4, and then terminate. So there are four possible final states ■

(b) Below are three derived variables. Indicate the *strongest* property from the list below that each variable is guaranteed to satisfy, no matter what the starting graph G is, and justify your answer. The properties are: *constant, strictly increasing, strictly decreasing, weakly increasing, weakly decreasing, none of these*

For any state, let e be the number of edges in it, and let c be the number of connected components it has. For example, since e may increase or decrease in a transition, it does not have any of the first four properties, so it would be classified *none of these*. The derived variables are

i) c ,

Solution. weakly decreasing ■

ii) $c + e$,

Solution. weakly decreasing ■

iii) $2c + e$.

Solution. strictly decreasing ■

(c) Conclude that the procedure terminates.

Solution. Because the derived variable iii) is a natural-number valued variable and is decreasing. To show that the variable iii) decreases, note that the rule for deleting an edge ensures that the connectedness relation does not change, so neither does the number of connected components; therefore the variable decreases by one when an edge is deleted. The rule for adding an edge ensures that the number of connected components decreases by one, so the variable changes by -2 for the reduced number of components and $+1$ for the added edge, for a net decrease of 1. ■

(d) Prove that any final state must be a tree on the vertices.

Solution. We use the characterization of a tree as a cycle-free, connected, simple graph.

A final state must be connected, because otherwise there would be two vertices with no path between them, and then a transition adding the edge between them would be possible, contradicting finality of the state.

A final state can't have a cycle, because deleting any edge on the cycle would be a possible transition. ■

Problem 2. The table below shows the preferences of each girl and boy in decreasing order.

<i>boys</i>	<i>girls</i>
1 : <i>CBEAD</i>	<i>A</i> : 35214
2 : <i>ABECD</i>	<i>B</i> : 52143
3 : <i>DCBAE</i>	<i>C</i> : 43512
4 : <i>ACDBE</i>	<i>D</i> : 12345
5 : <i>ABDEC</i>	<i>E</i> : 23415

(a) We saw that using a “boy greedy” strategy, where each boy in turn got his favorite available girl led to marriages in which Boy 4 and Girl C are a rogue couple. Which other boys are in rogue couples in these marriages? (You should reconstruct the greedy marriages from the table.)

Solution. The boy greedy marriages were:

$$\begin{aligned} 1 &\rightarrow C \\ 2 &\rightarrow A \\ 3 &\rightarrow D \\ 4 &\rightarrow B \\ 5 &\rightarrow E \end{aligned}$$

Boy 5 and Girl B are a rogue couple; so are Boy 5 and Girl A.

But Boys 1, 2, and 3 all got their top pick among the girls; none would even think of running off.

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(b) Find a stable set of marriages. (You can do this by trial and error; you're not expected to remember the Mating Procedure from the Notes, which we'll review later in class.)

Solution. Since all girls have different first choices, a "girl greedy" strategy will immediately instantly lead to stable marriages. The Mating algorithm yields another set.

$$\begin{aligned} 1 &\rightarrow E \\ 2 &\rightarrow B \\ 3 &\rightarrow D \\ 4 &\rightarrow C \\ 5 &\rightarrow A \end{aligned}$$

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Problem 3. Four Students want separate assignments to four VI-A Companies. Here are their preference rankings:

Student	Companies
Albert:	HP, Bellcore, AT&T, Draper
Carole:	AT&T, Bellcore, Draper, HP
Eric:	HP, Draper, AT&T, Bellcore
Radhi:	Draper, AT&T, Bellcore, HP

Company	Students
AT&T:	Radhi, Albert, Eric, Carole
Bellcore:	Eric, Carole, Albert, Radhi
HP:	Radhi, Eric, Albert, Carole
Draper:	Carole, Radhi, Eric, Albert

Use the Mating Algorithm to find *two* stable assignments of Students to Companies.

Solution. Treat Students as Boys and the result is the following assignment:

Student	Companies	Rank in the original list
Albert:	Bellcore	2
Carole:	AT&T	1
Eric:	HP	1
Radhi:	Draper	1

Treat Companies as Boys and the result is the following assignment:

Company	Students	Rank in the original list
AT&T:	Albert	2
Bellcore:	Carole	2
HP:	Eric	2
Draper:	Radhi	2

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Problem 4. Prove that the Mating Algorithm produces stable marriages. (Don't look up the proof in the Course Notes.)

Solution. *Proof.* Let Bob be some boy and Carole some girl that he does *not* marry on the last day of the Mating ritual. We will prove that Bob and Carole are not a rogue couple. Since Bob was an arbitrary boy, it follows that no boy is part of a rogue couple. Hence the marriages on the last day are stable.

To prove the claim, we consider two cases:

Case 1. Carole is not on Bob's list. Then since invariant P holds, we know that Carole prefers her husband to Bob. So she's not going to run off with Bob: the claim holds in this case.

Case 2. Otherwise, Carole is on Bob's list. But since Bob is not married to Carole, he must have chosen to serenade his wife instead of Carole, so he must prefer wife. So he's not going to run off with Carole: the claim also holds in this case. □

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