Mathematics for Computer Science

MIT 6.042J/18.062J

Combinatorics II.2

The Magic Trick

1. The audience chooses 5 cards from a shuffled deck.
2. The assistant reveals 4 cards.
3. The magician correctly guesses the 5th card.

Q. What is the role of the assistant?

What Can the Assistant Do?

Decide in which *order* to reveal the cards:

- 4 cards $\Rightarrow$ 4! orders = 24 orders.

Decide *which* 4 cards to reveal:

- 5 ways to choose which card to hide.

Magic Trick Revealed (I)

The audience picks 5 cards:

∴ At least 2 cards must have the same suit (*Pigeonhole Principle*).

Aha! The first card has the same suit as the hidden card!

Magic Trick Revealed (II)

How do we figure out the value of the hidden card?

Aha! Look at the order of the other 3 cards!

Total order on the cards

A♣ < 2♣ < 3♣ < ⋯ < K♣
< A♦ < 2♦ < 3♦ < ⋯ < K♦
< A♥ < 2♥ < 3♥ < ⋯ < K♥
< A♠ < 2♠ < 3♠ < ⋯ < K♠

Magic Trick Revealed (II)

How do we figure out the value of the hidden card?

Aha! Look at the order of the other 3 cards!

Another possible order

A♠ < A♥ < A♦ < A♣
< 2♠ < 2♥ < 2♦ < 2♣
⋮
< K♠ < K♥ < K♦ < K♣

Magic Trick Revealed (III)

But, wait!* *There are 12 possible values for the hidden card and only 6 permutations for the other 3 revealed cards.

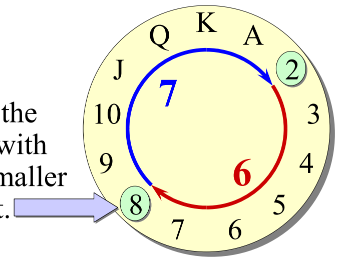
- Eliminating the value of the first card gives 13 – 1 = 12 values for the hidden card.
- Possible orders for the other 3 cards: { **SML**, **SLM**, **MSL**, **MLS**, **LSM**, **LMS** }

Aha! Of the two cards with the same suit, the choice of which is revealed gives 1 more bit of information!

Clockwise Distance

Between any 2 of the 13 values, the smaller clockwise distance is at most 6:

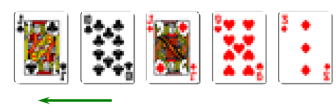
Hide the card with the smaller offset.

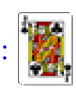
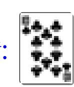


Magic Trick Revealed (Finally)

- The first card determines the suit (♠ ♥ ♦ ♣) of the hidden card.
- Hidden-card value = first-card value + offset (≤ 6).
- The offset is determined by the permutation of the other three cards: **SML** = 1, **SLM** = 2, **MSL** = 3, **MLS** = 4, **LSM** = 5, **LMS** = 6.

Example



Hidden:  *First*: 

Offset = 1 = **SML**: 

In-Class Problems

Problem 1

Why the Magic Trick Cannot Work with Only 4 Cards:

Audience can pick any 4-card combination:

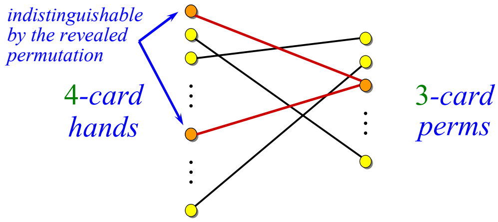
$$C(52,4) = \binom{52}{4} = 270{,}725$$

Assistant can reveal a 3-card permutation:

$$P(52,3) = \binom{52}{3} \cdot 3! = 132{,}600$$

$$C(52, 4) > P(52, 3)$$

Combinatorial Argument



Since $C(52, 4) > P(52, 3)$, by the Pigeonhole Principle, two 4-card hands must map to the same 3-card permutation.

5-Card Magic Trick

Audience picks a 5-card combination:

$$C(52,5) = \binom{52}{5} = 2{,}598{,}960$$

Assistant picks a 4-card permutation:

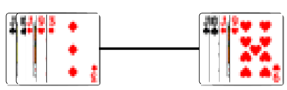
$$P(52,4) = \binom{52}{4} 4! = 6{,}497{,}400$$

Combinatorics do not rule out this trick.

But, can we always find a consistent mapping between hands and sequences?

Matching 5-Card Hands with 4-Card Permutations

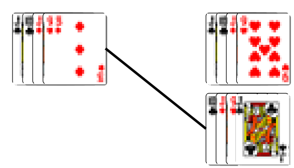
5-card hands (order does not matter)



4-card perms (order matters)

Matching 5-Card Hands with 4-Card Permutations

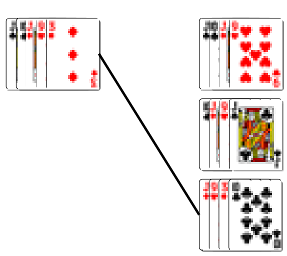
5-card hands (order does not matter)



4-card perms (order matters)

Matching 5-Card Hands with 4-Card Permutations

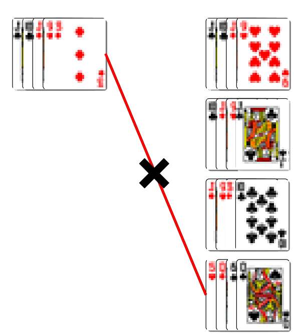
5-card hands (order does not matter)



4-card perms (order matters)

Matching 5-Card Hands with 4-Card Permutations

5-card hands (order does not matter)

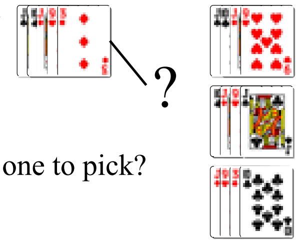


4-card perms (order matters)

Matching 5-Card Hands with 4-Card Permutations

5-card hands (order does not matter)

4-card perms (order matters)

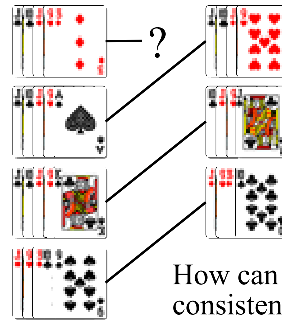


Which one to pick?

Matching 5-Card Hands with 4-Card Permutations

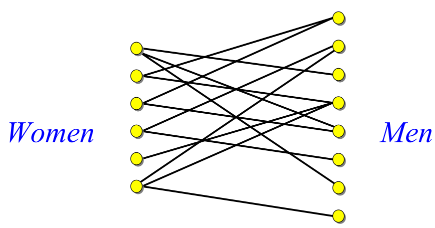
5-card hands (order does not matter)

4-card perms (order matters)



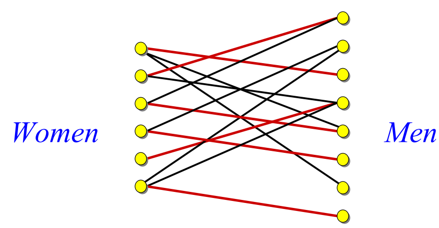
How can we ensure consistency?

Perfect Matchings



In a bipartite graph $G = (V_1, V_2, E)$, a ***perfect matching*** is a set $M \subseteq E$ such that every vertex in V_1 is incident on exactly one edge in M and every vertex in V_2 is incident on at most one edge in M.

Perfect Matchings

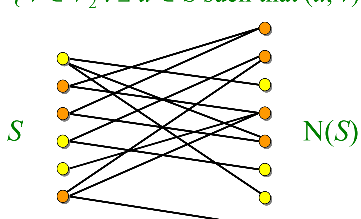


In a bipartite graph $G = (V_1, V_2, E)$, a ***perfect matching*** is a set $M \subseteq E$ such that every vertex in V_1 is incident on exactly one edge in M and every vertex in V_2 is incident on at most one edge in M.

Neighbor Sets

Definition. For a subset $S \subseteq V_1$, let $\mathrm{N}(S)$ denote the set of S's ***neighbors*** in V_2:

$$\mathrm{N}(S) = \{ v \in V_2 : \exists\, u \in S \text{ such that } (u, v) \in E \} .$$

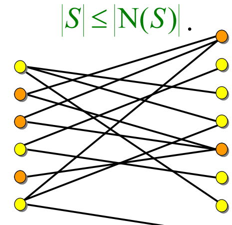


Hall's Marriage Theorem

Theorem. A perfect matching exists in a bipartite graph $G = (V_1, V_2, E)$ if and only if for every subset $S \subseteq V_1$, we have

$$|S| \le |\mathrm{N}(S)| .$$

$|S| = 3$



$|\mathrm{N}(S)| = 2$

No perfect matching exists.