

## Solutions to In-Class Problems — Week 2, Wed

**Problem 1. (a)** Prove that a  $2^n \times 2^n$  plaza can be tiled using L-shaped tiles leaving un tiled *any* single  $1 \times 1$  square desired. (Yes, the solution was in the reading, but do you remember it, and can you explain/write it?)

**Solution.** This variation of the problem is proved in the text, *Rosen pp. 196–197* and in [Week 2 Notes](#). ■

**(b)** If you finish part (a) fast, you can think about what happens if the area of the plaza is not a power of two. For example, can you characterize exactly which  $m \times n$  rectangles can be L-tiled with a single empty square where? (We don't know a complete classification of the L-tilable rectangles.)

**Problem 2.** Pinpoint, and illustrate with a counterexample, *exactly* where the following proof goes awry.

**False Claim.**  $\forall n \in \mathbb{N} \forall x \in \mathbb{R} x^n = 1$ .

*False proof.* (by induction on  $n$ )

**Induction Hypothesis:**

$$P(n) ::= \forall k \leq n \forall x \in \mathbb{R} x^k = 1.$$

**Base case** ( $n = 0$ ):  $P(0)$  is  $\forall x \in \mathbb{R} x^0 = 1$ . This holds by definition of “zeroth power” of a real number.

**Induction step:** Assuming  $P(n)$ , we have  $x^k = 1$  for all  $k \leq n$ . To prove  $P(n + 1)$  we must show that  $x^k = 1$  for all  $k \leq n + 1$ . Since the induction hypothesis already handles  $k \leq n$ , we need only show that  $x^{n+1} = 1$ . But applying the induction hypothesis with  $k = n$  and  $k = 1$ , we have  $x^n = 1$  and  $x^1 = 1$ . So,

$$x^{n+1} = x^n x^1 = 1 \times 1 = 1.$$

□

**Solution.** The argument that  $P(n)$  implies  $P(n + 1)$  is not correct when  $n = 0$ .

To see why this is so, let  $n = 0$ . So the hypothesis  $P(n)$  becomes  $\forall k \leq 0 x^k = 1$ . But since  $k$  is a natural-number valued variable,  $k \leq 0$  means  $k = 0$ , and so the hypothesis simplifies to  $x^0 = 1$ , which is true. But in addition to assuming that  $x^k = 1$  when  $k = n$ , the false proof above also requires that  $x^k = 1$  when  $k = 1$ . We can't assume this from the induction hypothesis because it is not true that  $k \leq n$  when  $k = 1$  and  $n = 0$ . ■

**Problem 3.** Prove by induction on the size of finite sets,  $A, B$ , that

$$|A \cup B| = |A| + |B| - |A \cap B| \quad (1)$$

**Solution.** Equation (1) makes no mention of an induction variable  $n$ , so to apply induction, we have to reformulate it in terms of  $n$ . There are several reasonable ways to do this. We'll choose

$$P(n) ::= \forall A |A| = n \longrightarrow \forall B |A \cup B| = |A| + |B| - |A \cap B|.$$

Now the conclusion of an induction proof will be  $\forall n P(n)$ . This is not quite the same as (1)—it mentions  $n$ —but is obviously equivalent to (1). In other words, all we have to do is prove  $\forall n P(n)$ .

*Proof.* (by Induction). The induction hypothesis,  $P(n)$ , was defined above.

We first establish:

**Lemma.** If  $A \subseteq B$ , then (1) holds.

*Proof.* (of Lemma). In this case,  $A \cup B = B$ ,  $A \cap B = A$ , so equation (1) reduces to

$$|B| = |A| + |B| - |A|,$$

which holds trivially. □

We now proceed with the induction proof.

**Base case:**  $n = 0$ . This means  $A = \emptyset$ . So surely,  $A \subseteq B$ , and equation (1) holds by the Lemma. So  $P(0)$  is true.

**Induction step:** Assume the induction hypothesis,  $P(n)$ , holds in order to prove  $P(n + 1)$ .

So suppose  $|A| = n + 1$ . We want to prove that (1) holds for all finite sets  $B$ . To do this we proceed by cases. Let  $B$  be any given finite set.

Case 1:  $A \subseteq B$ . Then (1) holds by the Lemma. (Note: we didn't even need to assume  $P(n)$ .)

Case 2:  $A \not\subseteq B$ . In this case, there is an  $a \in A - B$ . Let  $A' ::= A - \{a\}$ . So  $|A'| = n$ , and from  $P(n)$ , we have

$$|A' \cup B| = |A'| + |B| - |A' \cap B| \quad (2)$$

Now

$$A \cup B = (A' \cup \{a\}) \cup B = \{a\} \cup (A' \cup B),$$

and since  $a \notin (A' \cup B)$ , we have

$$|A \cup B| = |\{a\} \cup (A' \cup B)| = 1 + |(A' \cup B)|. \quad (3)$$

Also,  $A \cap B = A' \cap B$ , so from (2) we have

$$|A' \cup B| = n + |B| - |A \cap B|. \quad (4)$$

Now from (3),

$$\begin{aligned} |A \cup B| &= 1 + |A' \cup B| \\ &= 1 + n + |B| - |A \cap B| \\ &= |A| + |B| - |A \cap B|. \end{aligned}$$

So we have proved (1) in this case also.

Hence (1) holds for all  $B$ , and we have completed the proof of  $P(n+1)$ . □

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